A STUDY OF PROBABILISTIC METRIC SPACES

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Dedicated To
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CERTIFICATE

This is to certify that dissertation entitled *"A study of probabilistic metric spaces"* has been completed by *Mr. Mohd Tanveer* under my guidance. This work is more than adequate for the partial fulfillment for the award of the degree of Master of Philosophy in Mathematics.

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PREFACE

The purpose of this dissertation is to give a comprehensive introduction to the study of probabilistic metric spaces and non linear operator theory in probabilistic metric spaces. In 1942, K. Menger [28] introduced the concept of probabilistic metric spaces and by now a number of authors have done considerable works on probabilistic metric spaces. While writing this dissertation, we are greatly benefitted by the monographs of 'B. Schweizer and A. Sklar' (title: Probabilistic Metric Spaces), 'S. S. Chang, Y. J. Cho, and S. M. Kang' (title: Probabilistic Metric Spaces and Non Linear Operator Theory) and 'O. Hadžić and E. Pap' (title: Fixed Point Theory in Probabilistic Metric Spaces) which are proving very handy references in this domain. In the course of writing, we are greatly facilitated by their splendid efforts.

The study of probabilistic metric spaces was initiated by Schweizer and Sklar [44] and some of their coworkers. The history of probabilistic metric spaces is brief. In the original paper, Menger gave postulates for the distribution functions $F_{p,q}$, which included a generalized triangle inequality. In addition, he constructed a theory of betweenness and indicated possible fields of application. In 1943, shortly after the appearance of Menger's paper, Wald [64] published a paper in which he criticized Menger's generalized triangle inequality and proposed an alternative one. On the basis of this new inequality, A. Wald [65] constructed a theory of betweenness having certain advantages over Menger's theory. In 1951, Menger [29] continued his study of probabilistic metric spaces in a paper devoted to a resume of the earlier work, the construction of several specific examples and further considerations of the possible applications of the theory.

This dissertation consists of four chapters. Chapter 1 deals with the notion of probabilistic metric spaces with particular emphasis on the triangle inequality, the construction of particular spaces and the notion of topology, convergence and continuity in probabilistic metric spaces. Furthermore, we discuss the concept and geometric properties of Wald spaces and the relations between Wald spaces and probabilistic metric spaces. The most of the results of this chapter are available in Rhodes [41], B. Schweizer, A. Sklar and E. Thorp [42], [45], [48], H. Sherwood [53, 55] and others.

Chapter 2 is mainly devoted to various properties of PM-spaces with a special emphasis on topological structures, the problem of metrization and the form of metric functions for PM-spaces followed by a discussion on Menger PM-spaces and probabilistic normed linear spaces along with various characterizations of probabistically bounded sets. As applications, these results are utilized to study the linear operator theory on PM-spaces. In the same continuation, the concepts of topology, uniformity, metrics, semi-metrics (along with functions of non-compactness) are discussed. Since 1960 onwards, various aspects of theory and applications of PM-spaces have been developed by many authors especially Chang et al. [4-8,10], Hadžić [14, 15, 17], Hicks et al. [19, 21, 22], Radu [34-38], Schweizer and Sklar [45, 47, 48], Sherwood [53-55] and others which include results on the topological structures and properties for PM-spaces, the metrization problem and the form of metric functions in PM-spaces, Menger PM-spaces and PN-spaces. The several questions regarding characterizations of various probabilistically bounded sets in the linear operator theory and fixed point theory in PM-spaces remain unanswered. The purpose of this chapter is to investigate relevant results on these related topics.

In Chapter 3, we attempt an exhaustive survey of the fundamental results in the theory of triangular norms (see [26]) including some new results closely related to the theory of the fixed point in PM-spaces. The recent development of the theory of t-norms, (available in [26]), has a strong influence to the state of fixed point theory in PM-spaces. The first section of the chapter is devoted to basic definitions and some important examples of triangular norms and co-norms. In Section 3.2, we present some basic properties of triangular norms. The ordinal sum construction of t-norms is given in Section 3.3 whereas Section 3.4 is devoted to the representation of continuous t-norms. Especially important classes of t-norms with left continuous diagonals in the point $(1, 1)$ and t-norms of H-type are given in Sections 3.6 and 3.7, respectively. Section 3.8 is devoted to the practical conditions for comparison of t-norms with respect to the pointwise order as well as the domination relation between t-norms, which is important for the product of PM-spaces. Countable infinitary comparison of t-norms with respect to special classes of t-norms is investigated in Section 3.10.

In Chapter 4, fixed point theory in PM-spaces (only selected ones) are discussed which is a part of Probabilistic Analysis and continues to be an active area of mathematical research. Fixed point theory in PM-spaces is still in its formative stage. We anticipate that in near future we shall witness the development of a unified theory of random equations that will be based, partly, on concepts and tools of the fixed point theory in PM-spaces. Sherwood [54] showed that for a very large class of triangular norms it is possible to construct complete Menger PM-spaces admitting contraction mappings without fixed points. Recently, Hicks [19] considered another notion of contraction mappings namely H-contractions and showed that every such a mapping on a complete Menger PM-space $(X, \mathcal{F}, \min)$ has a unique fixed point. We conclude this chapter with a relatively recent fixed point theorem due Singh and Jain [58] which generalizes some earlier results due to Mishra [30], Hadžić [16] and Singh and Pant [59] and others.

This dissertation concludes with the bibliography which by no means is an exhaustive one but lists only those monographs, books and research papers which are referred to in this exposition.

CHAPTER 1

AN INTRODUCTION TO PROBABILISTIC METRIC SPACES

1.1 Introduction

Functional analysis plays a crucial role to bridge pure and applied sciences. In fact, it plays a key role in solving the different type of problems of varied type via fruitful interactions throughout. Indeed the evolution of functional analysis has greatly influenced the development of mathematics in recent years.

In Analysis and Algebra one is generally concerned with various kind of spaces (e.g. vector spaces, metric spaces etc.) where the usage of the word 'space' is somewhat confusing due to the literary meaning of the word space. In Mathematics by space one generally means a set together with a suitable set of conditions which are generally framed to abstract some natural situation. These abstract spaces are of basic importance and are employed to describe very wide and surprisingly general situations. In all, an abstract space will be a set of (unspecified) elements satisfying certain set of axioms and henceforth by varying sets and axioms different types of abstract spaces are defined in (abstract) mathematics.

The term metric seems to have derived from the word '*metor*' (measure). The most fruitful notion of metric spaces was introduced by French mathematician M. Fréchet in his Ph.D. thesis presented to Paris University in 1906. Though the definition presently in use, is the one formulated by the German mathematician F. Housdorff in 1914, yet the credit to introduce this fundamental and basic notion is entirely with Fréchet.

It will not be an exaggeration if one mentions that there could be no fruitful mathematics without metric and practically speaking, it is even present in school mathematics though a little indirectly and what to say of overdependence of any calculus on metric. For example in elementary calculus, we study functions defined on the real line R is equipped with natural distance though we never mention it at intermediate level.

The concept of an abstract metric space presents a natural frame of a large number of mathematical, physical and other scientific constructs in which the notion of a "distance", appears. The objects under consideration may be of most varied type. They may be points, functions, sets and even the subjective experiences of sensations. What turns out to be very sweet is the possibility of associating a non-negative real number with each ordered pair of elements of a certain set, and that the numbers associated with pairs and triples of such elements satisfy certain natural axioms patterned after natural distance.

In functional analysis we study more general 'spaces' and 'functions' defined on them wherein we replace the set of real numbers R by an abstract set X whose nature of elements remain unspecified and associate a distance function with X enabling one to measure distances between two arbitrary points of the set X. The formal but rigorous definition of metric space runs as follows.

Definition 1.1.1 A metric space is an ordered pair (X,d), where X is an abstract set and d is a mapping of $X \times X$ into R, i.e., d associates a real number $d(p,q)$ with every pair (p,q) of elements of X. The mapping d is assumed to satisfy the following conditions:
(M-1) $d(p,q)=0$ if and only if $p=q$, (identity)
(M-2) $d(p,q)\geq 0$, (Positivity)
(M-3) $d(p,q)=d(q,p)$ (Symmetry)
(M-4) $d(p,r)\leq d(p,q)+d(q,r)$. (Triangle inequality)

However, in numerous instances in which the theory of metric spaces is applied, this very association of a single number with a pair of elements is, realistically speaking, an over-idealization. This is so even in the measurement of an ordinary length, where the number given as the distance between two points is often not the result of a single measurement, but the average of a series of measurement. Indeed, in this and in many similar situations, it is appropriate to look upon the distance concept as a statistical measure rather than a determinate one. More precisely, he replaced the metric function $d: R\times R \longrightarrow R^{+}$ with a distribution function $F_{p,q}: R \longrightarrow [0,1]$ and then for any positive number x , $F_{p,q}(x)$ is interpreted as the probability that the distance between p and q is less than x. This very give rise a generalization to the concept of a metric spaces which was first introduced by K. Menger [28] in 1942 which is generally called a statistical metric space.

The history of statistical metric spaces is brief. In the original paper, [28] Menger gave postulates for the distribution functions $F_{p,q}$, with a special emphasis a generalized triangle inequality. In addition, he constructed a theory of betweenness and indicated possible fields of application.

In 1943, shortly after the appearance of Menger's paper, A. Wald [64] criticized Menger's generalized triangle inequality and proposed an alternative one. On the basis of this new inequality, A. Wald [65] constructed a theory of betweenness having certain advantages over Menger's theory. K. Menger [29] continued his study of statistical metric spaces in a paper devoted to further consolidation of the earlier work, wherein he constructed several specific examples and also indicated further possible applications of the theory.

In this chapter, we begin with the axioms of probabilistic metric spaces with particular emphasis on the triangle inequality, the construction of particular spaces , a consideration of topological notions in probabilistic metric spaces and a study of the continuity properties of the distance functions. Secondly, we introduce the concept and geometric properties of Wald spaces and the relations between Wald spaces and probabilistic metric spaces. The most of the results of this chapter are available in F. Rhodes [41], Schweizer, Sklar and Thorp [42] , [45] , [48] , Sherwood [53], [55] and others.

1.2 Probabilistic metric spaces

While defining probabilistic metric spaces (briefly, a PM-space), formally one uses the notion of distribution functions which runs as follows.

Definition 1.2.1 A mapping $\mathcal{F} : R \longrightarrow R^+$ is said to be a **distribution function** if it is non decreasing and left continuous with $\inf \mathcal{F} = 0$ and $\sup \mathcal{F} = 1$.
A frequently used specific distribution function H defined as

$$H(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

Throughout this text generally we denote the set of all distribution functions by $\mathcal{D}$.

Definition 1.2.2 A PM-space(in the sense of Schweizer and Sklar [44]) is an ordered pair $(X, \mathcal{F})$, where X be a non empty set and $\mathcal{F}$ is a map $\mathcal{F} : X \times X \longrightarrow D$, i.e., F associates a distribution function $\mathcal{F}(p, q)$ with every pair $(p, q) \in X \times X$
We denote distribution function $\mathcal{F}(p, q)$ by $F_{p,q}$ .For the real argument x, The function $F_{p,q}$ satisfies the following properties:
PM-1 $F_{p,q}(x) = 1$ for all $x > 0 \Longleftrightarrow p = q$,
PM-2 $F_{p,q}(0) = 0$,
PM-3 $F_{p,q} = F_{q,p}$,
PM-4 If $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$, then $F_{p,r}(x + y) = 1$.

Remark 1.2.3 In view of the condition PM-2, which obviously implies that $F_{p,q}(x) = 0$ for $x \leq 0$ and condition PM-2, is equivalent to statement

$$p = q \Longleftrightarrow F_{p,q} = H.$$

Remark 1.2.4 The principal inconvenience attached to implication PM-4 (in Definition 1.2.2) is that it is vacuous in all spaces in which the functions $F_{p,q}$, for $p \neq q$, never attains the value 1.

Remark 1.2.5 Every metric space may be viewed as a PM-space (of a special kind) if we set

$$F_{p,q}(x) = H(x - d(p, q))$$

The condition PM-4 of above definition is always satisfied in a metric space where it reduces to the ordinary triangle inequality.

In order to define PM-spaces in the sense of Serstnev [51], formally one uses the notion of triangle function which runs as follows.

Definition 1.2.6 A triangle function Δ is a binary operation on D that is commutative, associative, and non-decreasing in each place, and has H_0 as identity.

Similarly as we have done for t-norms, we can compare triangle functions in the following way: If Δ_1 and Δ_2 are triangle functions then Δ_1 is weaker than Δ_2 (Δ_2 is stronger than Δ_1), $\Delta_1 \leq \Delta_2$, if for all F, G in D and all x in $\mathbb{R}^+$

$$\Delta_1 (F, G)(x) \leq \Delta_2 (F, G)(x).$$

Example 1.2.7 Let T be a left-continuous t-norm. Then the function $T : D \times D \to D$ defined by

$$T (F, G)(x) = T (F (x), G (x))$$

is a triangle function.

Example 1.2.8 T_M defined by

$$T_M (F, G)(x) = T_M (F (x), G (x))$$

is a maximal triangle function. Indeed for any triangle function Δ we have

$$\Delta (F, G) \leq \Delta (F, H_0) = F$$

and

$$\Delta (F, G) \leq \Delta (H_0, G) = G.$$

Hence

$$\begin{aligned} \Delta (F, G)(x) &\leq T_M (F (x), G (x)) \\ &= T_M (F, G)(x) \quad \left(x \in R^+\right). \end{aligned}$$

Example 1.2.9 If T be a left-continuous t-norm,then Δ_T defined by

$$\Delta_T (F, G)(x) = \sup \{T (F (u), G (v)) \mid u + v = x\}$$

is a triangle function. We can also write for a t-norm T and $F, G \in D$

$$\Delta_T (F, G)(x) = \sup_{\alpha,\beta\geq 0,\alpha+\beta=1} T (F (\alpha x), G (\beta x)) (x \in R). \tag{1.2.9.1}$$

A triangle function is continuous, if it is continuous in the topology of the weak convergence on D.

Definition 1.2.10 (1) A PM-space (in the sense of Serstnev [51]) is a triple $(X, \mathcal{F}, \Delta)$ where X is a non empty set, $\mathcal{F} : X \times X \to D$ is given by $(p, q) \mapsto F_{p,q}$ and Δ is a triangle function, such that the following conditions are satisfied for all p, q, r in X :

(i) $F_{p,p} = H_0$,
(ii) $F_{p,q} \neq H_0$ for $p \neq q$,
(iii) $F_{p,q} = F_{q,p}$,
(iv) $F_{p,r} \geq \Delta\left(F_{p,q}, F_{q,r}\right)$.
(2) $(X, \mathcal{F}, \Delta)$ is proper if

$$\Delta\left(H_a, H_b\right) \geq H_{a+b} \quad (a, b \in [0, \infty)).$$

In case conditions (i) and (iii) in preceding definition hold, then the pair $(X, \mathcal{F})$ is a probabilistic premetric space. whereas in the event of realization of conditions (i), (iii) and (iv) (in Definition 1.2.10), the triple $(X, \mathcal{F}, \Delta)$ is a probabilistic pseudo-metric space and $(X, \mathcal{F})$ is a probabilistic premetric space under Δ and if the conditions (i), (ii) and (iii) (in Definition 1.2.10) hold, then the pair $(X, \mathcal{F})$ is a probabilistic semi-metric space.

From now on whenever we say that "$(X, \mathcal{F})$ is a PM-space", we mean that $(X, \mathcal{F})$ is a PM-space in the sense of Schweizer and Sklar [44], whereas the triplet $(X, \mathcal{F}, \Delta)$ is a PM-space in the sense of Serstnev [51].

In [12], R. Egbert introduced the notion of the product of PM-spaces.
Let $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ be PM-spaces and Δ a triangle function. The Δ-product of $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ is the ordered pair $(M_1 \times M_2, \mathcal{F}_1 \Delta \mathcal{F}_2)$ such that $\mathcal{F}_1 \Delta \mathcal{F}_2$ is a function defined on $(X_1 \times X_2)^2$ and with values in D^+ defined by

$$(\mathcal{F}_1 \Delta \mathcal{F}_2)(p, q) = \Delta\left(\mathcal{F}_1(p, q), \mathcal{F}_2(p, q)\right)$$

$\forall\ p = (p_1, p_2)$ and $q = (q_1, q_2)$ form $X_1 \times X_2$.

Definition 1.2.11 A triangular inequality is said to hold universally in a PM-space iff it holds for all triples of points, distinct not in that space.
Let $\Delta : [0,1] \times [0,1] \longrightarrow [0,1]$ be a 2-place function which satisfy the following conditions:
(Δ-1) $0 \leq \Delta(a, b) \leq 1$,
(Δ-2) $\Delta(c, d) \geq \Delta(a, b)$, for $c \geq a, d \geq b$
(Δ-3) $\Delta\ (a, b) = \Delta\ (b, a)$,
(Δ-4) $\Delta(1, 1) = 1$,
(Δ-5) $\Delta(a, 1) > 0\ \ \forall a > 0$.

There can be numerous possible definitions for Δ. Here six of the simplest but natural 2-place functions are as follows:
$\Delta_1\ (a.b) = max\ (a + b - 1, 0)$,
$\Delta_2\ (a, b) = a.b$,
$\Delta_3\ (a, b) = min\ (a, b)$,
$\Delta_4\ (a, b) = max\ (a, b)$,

$\Delta_5\ (a,b) = a + b - ab,$
$\Delta_6\ (a,b) = min(a+b,1).$

Remark 1.2.12 K. Menger [28] introduced as generalized triangle inequality or Menger triangle inequality as
(PM-5) $\quad F_{p,r}(x+y) \geq \Delta\ (F_{p,q}(x)\,,\, F_{q,r}(y))\,, \quad \forall\ x,y \geq 0$
where Δ is a 2-place function which satisfy above mentioned condition Δ-1 to Δ-5
From condition Δ-4, one may notice that PM-5 contains the condition PM-4 as a special case.

Lemma 1.2.13 If a PM-space contains two distinct points, then the condition PM-5 can not hold universally in the space under the choice $\Delta_4 = \max$.

Proof: Let p and q be two distinct points of space and Let $x$and y satisfy $0 < y < x$. suppose that PM-5 holds universally with $\Delta_4 = max$. Then we have

$$F_{p,q}(x) \geq \max\ (F_{p,q}(x-y)\,,\, F_{q,q}(y)) = 1.$$

Since x can be any positive number, therefore by the condition PM-1, one gets $p = q$ which is a contradiction to our supposition. This completes the proof.

Lemma 1.2.14 If in a nonmetric PM-space, PM-5 holds universally for some choice of Δ satisfying the conditions Δ-1 to Δ-5, then the function Δ has the property that there exists a number a, $0 < a < 1$, such that $\Delta\ (a,1) \leq a$.

Proof: If PM-space is not a metric space , then there is a point $(p,q) \in X \times X$ for which $F_{p,q}$ assumes values other than 0 or 1. By the left continuity and monotonicity of $F_{p,q}$, this means that there is not merely a point but an open interval (x,y) on which we have $\quad 0 < F_{p,q} < 1$. Now assume that

$$\Delta\,(a,1) = a + \Phi\,(a)\,,$$

where $\Phi\,(a) \geq 0 \quad$ for $0 < a < 1$.
Let $z \in (x,y)$ and take $t > 0$. Then we have

$$\begin{aligned} F_{p,q}(z\ +t) &\geq \Delta\,(F_{p,q}(z)\,,\, F_{q,q}(t)) \\ &= \Delta\,(F_{p,q}(z)\,,\ 1) \\ &= F_{p,q}(z) + \Phi\,(F_{p,q}(z))\,. \end{aligned}$$

Now letting $t \to 0$, then we have

$$F_{p,q}(\digamma) \geq F_{p,q}(z) + \Phi\,(F_{p,q}(z)) \geq F_{p,q}(z)\,,$$

which proves the discontinuity of $F_{p,q}$ at z and hence at every point of (x,y)as z is arbitrarily chosen in (x,y). But this is a contradiction as a nondecreasing function, can be discontinuous at only denumerably many points. This completes the proof.

Theorem 1.2.15 If in a PM-space $(X,\mathcal{F},\Delta)$
(i) PM-5 holds universally.
(ii) Δ is continuous, then for any $x>0$,

$$\Delta\left(F_{p,q}(x), 1\right) \leq F_{p,q}(x).$$

Proof: Let p,q be in X and $x>0$ be given, if we choose y such that $0<y<x$, then we have

$$\begin{aligned} F_{p,q}(x) &\geq \Delta\left(F_{p,q}(x-y), F_{q,q}(y)\right) \\ &= \Delta\left(F_{p,q}(x-y), 1\right), \end{aligned} \tag{1.2.15.1}$$

letting $y\to 0^{+}$, we obtain,

$$F_{p,q}(x) \geq \lim_{y\to 0+} \Delta\left(F_{p,q}\ (x-y), 1\right). \tag{1.2.15.2}$$

Now, using the assumed continuity of Δ, one can have

$$\lim_{y\to 0+} \Delta\left(F_{p,q}(x-y), 1\right) = \Delta\left(\lim_{y\to 0+} F_{p,q}(x-y), 1\right) \tag{1.2.15.3}$$

which due to the left continuity of $F_{p,q}$, reduces to

$$\lim_{y\to 0+} F_{p,q}(x-y) = F_{p,q}(x),$$

now using (1.2.15.3) and (1.2.15.2), one gets

$$F_{p,q}(x) \geq \Delta\left(F_{p,q}(x), 1\right).$$

This completes the proof.

By these lemmas and the fact that the three weaker functions in the list of $\Delta^{'s}$ which satisfy $\Delta\ (a,1)=a$, we are led to replace the conditions Δ-1, Δ-4 & Δ-5 by the following conditions:
(Δ-6) : $\Delta\ (a,1)=a$ and $\Delta\ (0,0)=0$,
To the list of requirements to the function Δ, we also add the associativity condition
(Δ-7) : $\Delta\ (\ \Delta\ (a,b)\ , c) = \Delta\ (a,\ \Delta\ (b,c))$,
which allows the extension of Δ-6 to a polygonal inequality.
Now we are equipped to introduce the following definition.

Definition 1.2.16 A Menger PM-space is a PM-space in which the condition PM-5 holds universally for some choice of Δ satisfying Δ-2, Δ-3, Δ-6 and Δ-7.

Definition 1.2.17 A triangular norm (in short a t-norm) is a 2- place function $\Delta : [0,1] \times [0,1] \longrightarrow [0,1]$ satisfying the condition Δ-2, Δ-3, Δ-6 and Δ-7.

Since further discussions on t-norm disrupt the continuity of the subject matter, therefore we postpone this discussion for a which but keeping in view the importance of t-norm in PM-spaces, we intend to carry out a detailed study of triangular norm in Chapter 3.
The following lemma establishes the fact that, in determining whether on going a PM-space is a Menger PM-space , only triplet of distinct points are ought to be considered.

Lemma 1.2.18 If the points p, q, r are not all distinct , then the condition PM-5 holds for the triple p, q, r under any choice of Δ satisfying Δ-2, Δ-3, Δ-6 and Δ-7.

Proof: We need only consider the choice Δ_3 =min., If $p = r$, then $F_{p,r} = H$ and the conclusion is immediate. If $p = q \neq r$, then for any $x, y \geq 0$, one can write

$$\begin{aligned} min\left(F_{p,q}(x)\ ,\ F_{q,r}(y)\right) &= \min\left(H(x), F_{q,r}(y)\right) \leq F_{q,r}(y) \\ &\leq F_{q,r}(x+y) = F_{p,r}(x+y) \end{aligned}$$

which completes the proof.

1.3 Some specific classes of PM-spaces

In this section, we consider two important classes of PM-spaces namely "Menger spaces and Wald spaces".

Definition 1.3.1 Let $(X, \mathcal{F}, \Delta)$ be a PM-space and $\Delta = \Delta_T$, where

$$\Delta_T(F,G)(x) = \sup\left\{T\left(F(u), G(v)\right) \mid u + v = x\right\}$$

for a t-norm T. Then $(X, \mathcal{F}, \Delta)$ is called a **Menger space**, which will be denoted by $(X, \mathcal{F}, T)$ in the sequel.

Remark 1.3.2 If the t-norm Δ is left-continuous then Δ_T of Definition 1.3.1 is a triangle function. Then we have

$$F_{p,r}(x+y) \geq \Delta\left(F_{p,q}(x), F_{q,r}(y)\right) \tag{1.3.2.1}$$

$\forall\ p, q, r$ in X and x, y real numbers. This inequality also implies (iv). Namely taking $x \in [0, \infty)$ we have for all $u, v \in [0, \infty)$ such that $u + v = x$ with

$$F_{p,r}(x) \geq \Delta\left(F_{p,q}(u), F_{q,r}(v)\right).$$

Hence

$$F_{p,r}(x) \geq \left(\Delta_T\left(F_{p,q}, F_{q,r}\right)\right)(x).$$

We can interpret the inequality (1.3.2.1) in a similar way as one does in the classical metric spaces wherein the third side of the triangle depends on the other two sides in the sense that if the length of two sides increases then also the length of the third side increases or in other words knowing the upper bounds of the two sides we have an upper bound for the third side.

Remark 1.3.3 If the inequality

$$F_{x,z}(u+v) \geq \Delta\left(F_{x,y}(u), F_{y,z}(v)\right)$$

is replaced by the inequality

$$F_{p,r}(\max\ (x, y)) \geq \Delta\ \left(F_{p,q}(x), F_{q,r}(y)\right),$$

then the triplet $(X, \mathcal{F}, \Delta)$ is termed as **non-Archimedean Menger space.**

As a very special case of a Menger spaces, we obtain the classical metric space.

Example 1.3.4 If we suppose, that there exists a function $d : M \times M \longrightarrow [0, \infty)$, such that

$$F_{p,q}(x) = H_{d(p,q)} \quad (p, q \in M,\ x \in R), \tag{1.3.4.1}$$

then we find that the Menger space $(X, \mathcal{F}, \Delta_T)$ with $\mathcal{F}(p,q) = H_{d(p,q)}$, for any t-norm Δ, is a classical metric space. Naturally, we have to prove only the classical triangle inequality, since all other properties hold trivially. If we have for $p, q, r \in M$ with $d(p,q) < x$ and $d(q,r) < y$ for some $x, y. > 0$, then by 1.3.4.1 it follows that $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$. Then by (1.3.2.1) and the boundary property of t-norm Δ we obtain

$$d(p,r) < x + y,$$

which gives the desired inequality.

If we begin with a metric space (M, d), then by taking $F_{p,q}$ defined by 1.3.2.1 we find that for any t-norm Δ the function $F_{p,q}$ is a probability distribution function such that the condition (i) to (iv) in Definition 1.2.10 are satisfied for Δ_T. In this way we have proved that $(M, \mathcal{F}, \Delta_T)$ for $\mathcal{F}(p,q) = H_{d(p,q)}$is a Menger space if and only if (M, d) is a classical metric space.

Definition 1.3.5 (Wald space) The following inequality (generally referred as generalized inequality) is due *A-Wald* [65].
$(PM\text{-}w) \quad F_{p,r}(x) \geq [F_{p,q} * F_{q,r}](x) \quad \forall x \geq 0.$
where $*$ denotes convolution, i.e.

$$[F_{p,q} \, * \, F_{q,r}](x) = \int_{-\infty}^{+\infty} F_{p,q}(x-y)\, dF_{q,r}(y).$$

Since $F_{p,q}(x-y) = 0$ for $y \geq x$ and $F_{q,r}(y) = 0$ for $y \leq 0$,
we may evidently write

$$[F_{p,q} * \, F_{q,r}](x) = \int_{0}^{x} F_{p,q}(x-y)\, dF_{q,r}(y),$$

as the convolution of the distribution function of two independent random variables gives the distribution function of their sum.

Remark 1.3.6 The interpretation of $PM\text{-}w$ as the probability that the distance of $p \,\&\, r$ is less than x is not less than the probability that the sum of the distance of $p \,\&\, q$ and the distance of $q \,\&\, r$ (regarded as independent) is less than x.

Definition 1.3.7 A PM-space $(X, \mathcal{F}, \Delta)$ when Δ is a convolution is called a Wald space.

Using the equality $H_a * H_b = H_{a+b}$ and $F(p,q) = H_{d(p,q)}$ where $d : M \times M \to [0, \infty)$ one can show that the triple $(M, F, *)$ is a Wald space if and only if (M, d) is the classical metric space. Since the proof is analogous to the proof given for a general Menger PM-spaces, it is omitted.

Definition 1.3.8 A Wald space is a PM-space provided $PM\text{-}w$ holds universally.

Theorem 1.3.9 Every Wald space is a Menger PM-space under the choice $\Delta_2 =$ product.

Proof: In a Wald space, for any $x, y \geq 0$, we have

$$\begin{aligned} F_{p,r}(\,x+y\,) \;&\geq\; \int_{0}^{x+y} F_{p,q}(\,x+y-z\,)\, dF_{q,r}(z) \\ &= \int_{0}^{x+y} \left[\int_{0}^{x+y-z} dF_{p,q}\,(t) \right] dF_{q,r}(z) \\ &= \iint_{t,z\,\geq 0,\; t+z\,\leq\, x+y} dF_{p,q}(t)\,.dF_{q,r}(z)\,. \end{aligned}$$

Now,

$$\iint_{t,z\geq 0,\; t+z\leq x+y} dF_{p,q}(t)\,.dF_{q,r}(z) \geq \iint_{0\leq t\leq x,\; 0\leq z\leq y} dF_{p,q}(t)\,.dF_{q,r}(z),$$

as $\{(t,z)\,;\,0\leq t\leq x, 0\leq z\leq y\} \subset \{(t,z)\,;t,z\geq 0, t+z \leq x+y\}$
and the F are non decreasing. But

$$\begin{aligned}\iint_{0\leq t\leq x,\; 0\leq z\leq y} dF_{p,q}(t)\,.dF_{q,r}(z) &= \int_0^x\int_0^y dF_{p,q}(t)\,.dF_{q,r}(z)\,.\\ &= \int_0^x dF_{p,q}(t)\,.\int_0^y dF_{q,r}(z) = F_{p,q}(x)\,.F_{q,r}(y)\end{aligned}$$

therefore by combining the earlier inequalities, we obtain

$$F_{p,r}(x+y) \geq F_{p,q}(x)\,.F_{q,r}(y), \tag{1.3.9.1}$$

which is indeed PM-5 under $\Delta_2 =$ product.

Corollary 1.3.10 If the Wald inequality $PM\text{-}w$ holds, then so does the inequality PM-4.

Proof: Since a Wald space is a Menger PM-space in which $PM\text{-}w$ hold, by

$$F_{p,r}(x+y) \geq F_{p,q}(x)\,.F_{q,r}(y),$$

If $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$, then $F_{p,r}(x+y) = 1$.

Lemma 1.3.11 If the points p, q, r are not all distinct, then the condition $PM\text{-}w$ holds for the triple p, q, r.

Proof: If $p = r$, this is immediate, as in all case $F_{p,r} = H$, otherwise if $p = q \neq r$, then for $x \geq 0$,

$$\begin{aligned}F_{p,r}(x) &= F_{q,r}(x) = \int_0^x dF_{q,r}(y)\\ &= \int_0^x H(x-y)\,dF_{q,r}(y)\\ &= \int_0^x F_{p,q}(x-y)\,dF_{q,r}(y) \geq [F_{p,q} * F_{q,r}](x)\,.\end{aligned}$$

The case $p \neq q = r$ follows on interchanging r and p. This concludes the proof.

Theorem 1.3.12 If in a PM-space , the condition PM-5 holds for all triples of distinct points under $\Delta_4 = \max$, then the space is Wald space.

Proof: Let p, q, r be distinct, then for any $x \geq 0$, we have

$$\begin{aligned} F_{p,r}(x) &\geq \max\left(F_{p,q}(0), F_{q,r}(x)\right) = F_{q,r}(x) \\ &= \int_0^x dF_{q,r}(y) \geq \int_0^x F_{p,q}(x-y) \,.dF_{q,r}(y), \end{aligned}$$

as $0 \leq F_{p,q}(x-y) \leq 1$.Therefore the condition *PM-w* holds for the triple of distinct points in the space.
But in view of Lemma 1.3.11, *PM-w* holds automatically for triples of non distinct points, which amounts to say that *PM-w* holds for all triples of points in the space.

1.4 On some specific spaces

The simplest metric spaces are discrete spaces which are also referred as the equilateral spaces when the metric d is defined as

$$d(p,q) = \begin{cases} a, & p \neq q \\ 0, & p = q \end{cases}, \tag{1.4.1}$$

where a is positive. Analogously, we call a PM-space equilateral if for some distribution function G satisfying $G(0) = 0$,

$$F_{p,q}(x) = \begin{cases} G(x), & p \neq q \\ H(x), & p = q \end{cases}$$

where H is a distribution function described by Definition 1.2.1. It is straightforward to verify that all requirements (i.e., PM-1 $\rightarrow$ PM-4) of PM-spaces are satisfied.

Theorem 1.4.1 In an equilateral PM-space , the Menger triangle inequality (*i.e.*, PM-5) holds for any triple of distinct points under $\Delta_4 = \max$ and universally under $\Delta_3 = \min$.

Proof: Since G is non decreasing,

$$\begin{aligned} G(x+y) &\geq \max\left(G(x), G(y)\right) \\ &\geq \min\left(G(x), G(y)\right) \end{aligned}$$

and

$$G(x+y) \geq \min\left(G(x), 1\right).$$

Corollary 1.4.2 An equilateral PM-space is a Wald space.

Proof: This is a direct consequences of the last theorem. There do exists equilateral PM-spaces in which the condition PM-5 holds under a stronger choice of Δ. Now we give some examples which show that there are equilateral PM-spaces in which the generalized triangle inequality PM-5 holds under stronger Δ than $\Delta_4 = \max$.

Example 1.4.3 Let

$$G(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1, \\ 1 & 1 \leq x. \end{cases}$$

:For any triple of distinct points in this space, the condition PM-5 holds under $\Delta_6 = \min(\text{sum}, 1)$, as in all cases, we have

$$G(x+y) \geq \min\{G(x), G(y), 1\}$$

Example 1.4.4 Let

$$G(x) = \begin{cases} 0, & x \leq 0 \\ 1-e^{-x}, & x \geq 0. \end{cases}$$

For any triple of distinct points in this space, the condition PM-5 holds under $\Delta_5 =$ sum$-$ product. This is evident in view of the fact that $e^{-x}.e^{-y} = e^{-(x+y)}$.

A class of PM-space, which is more interesting than the equilateral PM-spaces. To define this, one needs the concept of specific distribution which runs as follows:
Let (X, d) be a metric space G be a distribution function, different from H, satisfying $G(0) = 0$. For every pair of points p, q in X, define the distribution function $F_{p,q}$ as follows:

$$F_{p,q}(x) = \begin{cases} G\left(\frac{x}{d(p,q)}\right) & p \neq q \\ H(x), & p = q \end{cases} \tag{1.4.4.1}$$

Definition 1.4.5 A PM-space $(X, \mathcal{F})$ is said to be a **simple space** iff there exists a metric d on X and a distribution function G satisfying $G(0) = 0$, such that for every points p, q in X, $\mathcal{F}_{p,q} = F(p,q)$ is given by (1.4.4.1). Furthermore, we say that $(X, \mathcal{F})$ is a simple space generated by the metric space (X, d) and the distribution function G.

Theorem 1.4.6 A simple space is a Menger PM-space under any choice of Δ satisfying Δ-2, Δ-3, Δ-6, and Δ-7.

Proof: It is sufficient to show that the condition PM-5 holds universally under $\Delta_3 = \min$, since this is the strongest choice of Δ possible. Thus from the Lemma 1.2.3, we have only to show that for p, q, r are distinct,

$$G\left(\frac{x+y}{d(p,r)}\right) \geq \min\left(G\left(\frac{x}{d(p,q)}\right), G\left(\frac{y}{d(q,r)}\right)\right). \tag{1.4.6.1}$$

Since d is an ordinary metric, therefore

$$d(p,r) \leq d(p,q) + d(q,r).$$

which inturn yields that

$$\frac{x+y}{d(p,r)} \geq \frac{x+y}{d(p,q) + d(q,r)} \tag{1.4.6.2}$$

Furthermore, since $d(p,q)$ and $d(q,r)$ are positive real numbers, therefore

$$\begin{aligned} \max\left(\frac{x}{d(p,q)}, \frac{y}{d(q,r)}\right) &\geq \frac{x+y}{d(p,q) + d(q,r)} \\ &\geq min\left(\frac{x}{d(p,q)}, \frac{y}{d(q,r)}\right) \end{aligned} \tag{1.4.6.3}$$

with the equality of either side iff

$$\frac{x}{d(p,q)} = \frac{y}{d(q,r)}.$$

Consequently, inequalities (1.4.6.2) and (1.4.6.3), we have

$$\frac{x+y}{d(p,r)} \geq min\left(\frac{x}{d(p,q)}, \frac{y}{d(q,r)}\right),$$

since G is non decreasing, implies (1.4.6.1). Which complete the proof.

Definition 1.4.7 Two metric spaces M_1, M_2 with distance function d_1 and d_2 respectively are said to be **homothetic** if there exists a number $a > 0$ and a one to one mapping f from M_1 to M_2 such that $d_1(p,q) = a\, d_2(f(p), f(q))$, $\forall p,q \in M_1$.

Corollary 1.4.8 An equilateral metric spaces generates an equilateral PM-space.

Corollary 1.4.9 If $G(x) = H(x-1)$, then generated PM-space reduces to the generating metric space.

Proof: Note that

$$F_{p,q}(x) = H\left(\frac{x}{d(p,q)} - 1\right) = H(x - d(p,q)).$$

In most simple spaces, Δ_4 =max will be too strong since inequality 1.4.6.3 shows that, for a triple of distinct points p, q, r with

$$d(p,r) = d(p,q) + d(q,r),$$

the condition PM-5 fails under $\Delta_4 = \max$.

1.5 Topology, Convergence and Continuity

Definition 1.5.1 Let p be a point in a PM-space $(X, \mathcal{F})$. By an (ϵ, λ) neighborhood of p, $\epsilon \geq 0$, $\lambda \geq 0$, we mean the set of all points q in X for which $F_{p,q}(\epsilon) > 1 - \lambda$, we write

$$N_p(\epsilon, \lambda) = \{q\colon\ F_{p,q}(\epsilon) > 1 - \lambda\}.$$

The interpretation is that $N_p(\epsilon, \lambda)$ is the set of all points q in X for which the probability of the distance from p to q is less than ϵ is greater than $1 - \lambda$. Observe that this neighborhood of a point in a PM-space depends on two parameters.

Theorem 1.5.2 In a simple space $N_p(\epsilon, \lambda)$ is an ordinary spherical neighborhood of p in a generating metric space.

Proof: For any p, q, we have

$$F_{p,q}(\epsilon) = G\left(\frac{\epsilon}{d(p,q)}\right),$$

which will be greater than $1 - \lambda$ provided only that $d(p,q)$ is sufficiently small.

Lemma 1.5.3 If $\epsilon_1 \leq \epsilon_2$ and $\lambda_1 \leq \lambda_2$, then $N_p(\epsilon_1, \lambda_1) \subset N_p(\epsilon_2, \lambda_2)$.

Proof: Suppose $q \in N_p(\epsilon_1, \lambda_1)$, so that

$$F_{p,q}(\epsilon_1) > 1 - \lambda_1.$$

Then we have

$$F_{p,q}(\epsilon_2) \geq F_{p,q}(\epsilon_1) > 1 - \lambda_1 \geq 1 - \lambda_2,$$

and so that by definition $q \in N_p(\epsilon_2, \lambda_2)$, which proves the result.

Definition 1.5.4 A sequence of points $\{p_n\}$ in a PM-space is said to converge to a point p in X. iff for every $\epsilon > 0$ and every $\lambda > 0$, there exists an integer $M_{\epsilon,\lambda}$ such that $p_n \in N_p(\epsilon, \lambda)$ i.e.

$$F_{p,p_n}(\epsilon) > 1 - \lambda \quad \text{whenever } n > M_{\epsilon,\lambda}.$$

Lemma 1.5.4 If $p_n \to p$, then $F_{p,p_n} \to F_{p,p} = H$, i.e.

$$\forall x, \quad F_{p,p_n}(x) \to F_{p,p}(x) = H(x).$$

and conversely.

Proof: (a) If $x > 0$, then for every $\lambda > 0$, there exists an integer $M_{x,\lambda}$ such that $F_{p,p_n}(x) > 1 - \lambda$, whenever $n > M_{x,\lambda}$. This means that

$$\lim_{n \to \infty} F_{p,p_n}(x) = 1 = F_{p,p}(x).$$

(b) If $x = 0$, then $\forall\, n$, $F_{p,p_n}(0) = 0$ and hence

$$\lim_{n \to \infty} F_{p,p_n}(0) = F_{p,p}(0).$$

The converse part is immediate.

Corollary 1.5.5 The convergence is uniform on $[a, b]$ such that $a > 0$, i.e., $M_{x,\lambda}$ is independent of x for $a \leq x \leq b$.

Proof: For any x, $a \leq x \leq b$,

$$F_{p,p_n}(x) \geq F_{p,p_n}(a).$$

Theorem 1.5.6 If $(X, \mathcal{F})$ is a Menger PM-space and Δ is continuous, then the probabilistic distance function F, is lower semi continuous function of points, i.e., for every fixed x, if $q_n \longrightarrow q$ and $p_n \longrightarrow p$. then

$$\inf \lim_{n \to \infty} F_{p_n,q_n}(x) = F_{p,q}(x).$$

Proof: For the proof, one may consult Chang et. al [9].

1.6 Geometric properties of PM-spaces

Definition 1.6.1 A Wald space is said to be convex if there is always a point between two distinct points .

Definition 1.6.2 A. Wald [65] introduced the concept of 'betweenness' as follows:
(i) If pqr, then rqp,
(ii) If pqr, then neither qrp nor rpq can hold,
(iii) If pqr and prs, then pqs and qrs.
where pqr is used to denote that q lies between p and r.

Proposition 1.6.3 If $\{p_n\}$ and $\{q_n\}$ are sequences of points of a Wald space X which converge, respectively, to p and q and if the distribution function $x \to F_{p,q}(x)$ is continuous at $x = \epsilon$, then $F_{p_n,q_n}(\epsilon) \to F_{p,q}(\epsilon)$ as $n \to \infty$, i.e., the mapping $(u, v) \to F_{u,v}(\epsilon)$ of $X \times X$ to the unit interval is continuous at (p, q).

Proof: For the proof ,one may consult Chang et. al [9].

Lemma 1.6.4 If X is a convex Wald space whose distribution functions are regular at ϵ, then for every pair of distinct points p and q, $F_{p,q}(x) < 1$ for all $x \geq \epsilon$, and if pqr, then $F_{q,r}(x) > F_{p,r}(x)$ for all x such that $0 \leq x \leq \epsilon$.

Proof: Since X is convex, there is a point s between p and q and since $p \neq s$, there exists a positive number δ less than ϵ , such that $F_{p,s}(x) < 1$ and for all $x \leq \delta$. Now we have

$$\begin{aligned} F_{p,q}(\epsilon) &= [F_{p,s} * F_{s,q}](\epsilon) \\ &= \int_0^{\epsilon} F_{p,s}(\epsilon - y) .dF_{s,q}(y) . \\ &= \int_0^{\epsilon-\delta} F_{p,s}(\epsilon - y)\, dF_{s,q}(y) + \int_{\epsilon-\delta}^{\epsilon} F_{p,s}(\epsilon - y)\, d\,F_{s,q}(y) \\ &< F_{s,q}(\epsilon - \delta) + \{F_{s,q}(\epsilon) - F_{s,q}(\epsilon - \delta)\} = F_{s,q}(\epsilon) \leq 1. \end{aligned}$$

Hence for all x for which $0 \leq x \leq \epsilon$,

$$F_{p,r}(x) = \int_0^{x} F_{p,q}(x-y)\, dF_{q,r}(y) < F_{q,r}(x)$$

which completes the proof.

Proposition 1.6.5 If X is a convex, compact Wald space whose distribution functions are regular at ϵ, then one parameter collection of neighborhoods $\zeta = \{U(\epsilon, \lambda)\,;\, \lambda > 0\}$ is a basis for the Housdorff uniformly on X.

Proof: For the proof, one may consult Chang et.al [9].

Lemma 1.6.6 Let X be a convex, compact Wald space whose distribution functions are regular at ϵ. If $F_{p,r}(\epsilon) = 1 - \lambda$ and $\mu < \lambda$, then there is a point q of X between p and r such that

$$F_{p,q}(\epsilon) = 1 - \mu.$$

Proof: Consider the set P of points q such that pqr and $F_{p,q}(\epsilon) < 1 - \mu$. Since the function $q \longrightarrow F_{p,q}(\epsilon)$ defined on the space X is continuous, in case there is no point q with $F_{p,q}(\epsilon) = 1 - \mu$, the set $P \cup \{r\}$ is closed and therefore compact. Hence the supremum of the function $q \rightarrow F_{p,q}(\epsilon)$ on the set $P \cup \{r\}$ is attained at some point s of the set, and $F_{p,s}(\epsilon) < 1 - \mu$.

Similarly, the set Q of points q such that pqs and $F_{p,q}(\epsilon) > 1 - \mu$ is non empty and $Q \cup \{p\}$ is compact, hence the function $q \to F_{p,q}(\epsilon)$ defined on the set $Q \cup \{p\}$ attains its infimum at some point t of the set t of the set, and

$$F_{p,t}(\epsilon) > 1 - \mu.$$

Consider any point u between s and t. Since psr, pts and tus, also pus and ptu,therefore

$$F_{p,u}(\epsilon) > F_{p,s}(\epsilon) = \sup F_{p,q}(\epsilon)$$

for points q such that pqr and $F_{p,q}(\epsilon) < 1 - \mu$, so that $F_{p,u}(\epsilon) > 1 - \mu$. Also we have

$$F_{p,u}(\epsilon) < F_{p,t}(\epsilon) = \inf F_{p,q}(\epsilon)$$

for points q such that pqs and $F_{p,q}(\epsilon) < 1 - \mu$, so that

$$F_{p,u}(\epsilon) < 1 - \mu.$$

which is a contradiction and hence there is a point q between p and r such that

$$F_{p,q}(\epsilon) = 1 - \mu.$$

This completes the proof..

Corollary 1.6.7 A subspace of a Wald space consisting of a pair p, r and all points between them is closed.

Proof: For the proof, one may consult Chang et.al [9].

Lemma 1.6.8 Let X be a convex, compact Wald spaces whose distribution functions are regular at ϵ. Given any neighborhood $U(\epsilon, \lambda)$, X contains for every two distinct points p and r, a point q such that pqr and $(p,q) \in U(\epsilon, \lambda)$.

Proof: Let Q be the set of all points between p and r. It is not null. Since $p \neq r$. and space X is convex. If, for a given neighborhood $U(\epsilon, \lambda)$, there is no point q of Q such that $(p,q) \in U(\epsilon, \lambda)$, then $F_{p,q}(\epsilon) \leq 1 - \lambda$ for all such points. Hence $p \notin \bar{Q}$ and it follows from Corollary 1.6.7 that

$$\bar{Q} \cup \{r\} = Q \cup \{r\}.$$

This is a closed subset of a compact Housdorff space and so is itself compact. From Proposition 1.6.3 and the condition that all the distribution function $x \to F_{u,v}(x)$ are continuous at $x = \epsilon$, it follows that the function $q \to F_{p,q}(\epsilon)$ defined on the compact set $\bar{Q}$ is continuous and so attains its supremum at a point s. Suppose

$$F_{p,s}(\epsilon) = 1 - \mu.$$

Then $1-\mu \leq 1-\lambda < 1$, so that $p \neq s$ and the convex space X contains a point t between p and s, then by property (iii) of 'between' relationship also lies between p and r. By Lemma 1.6.1,

$$F_{p,t}(\epsilon) > 1-\mu$$

and $1-\mu$ is not the supremum of $F_{p,q}(\epsilon)$ for points q between p and r. Hence given any two points p and r there are points p and r between lying arbitrarily close to p.

Proposition 1.6.9 If X is a convex, compact Wald space whose distribution functions are regular at ϵ, then given any pair of points (p, r), there is a point q such that pqr and $F_{p,q}(\epsilon) = F_{p,r}(\epsilon)$.

Proof: The function $q \rightarrow \mid F_{p,q}(\epsilon) - F_{q,r}(\epsilon) \mid$ is continuous and so attains its infimum at some point s, If the infimum is not zero, we may suppose that

$$F_{p,s}(\epsilon) - F_{s,r}(\epsilon) = 2m > 0.$$

Lemma 1.6.8 shows that points can be chosen between s and r and arbitrarily close to s, there is a point t such that t lies between s and r, therefore also between p and r, and

$$0 < F_{p,s}(\epsilon) - F_{p,t}(\epsilon) < m, \quad 0 < F_{t,r}(\epsilon) - F_{s,r}(\epsilon) < m.$$

Hence we have

$$F_{p,t}(\epsilon) - F_{t,r}(\epsilon) < F_{p,s}(\epsilon) - F_{s,r}(\epsilon) < F_{p,t}(\epsilon) - F_{t,r}(\epsilon) + 2m.$$

Since $F_{p,s}(\epsilon) - F_{s,r}(\epsilon)$ is the infimum of $\mid F_{p,q}(\epsilon) - F_{q,r}(\epsilon) \mid$ for all points q between p and r , $F_{p,t} - F_{t,r}$ must be negative. Hence it follows that

$$(F_{p,s}(\epsilon) - F_{s,r}(\epsilon)) < 2m.$$

This contradiction shows that infimum is zero which completes the proof.

Lemma 1.6.10 Let X be convex, compact Wald space whose distribution functions are regular at ϵ. If $\{q_n\}$ is a sequence of points of X such that $q_0q_{n+1}q_n$ and $F_{q_{0},q_{n+1}}(\epsilon)$ for all n, then $\{q_n\}$ converges to q_0.

Proof: The sequence $\{q_n\}$ certainly has a point of accumulation q, and since $q_0q_{n+1}q_n$ for all n, the sequence converges to q. If q is not q_0, then $F_{q_0,q}(\epsilon) = 1-\mu < 1$, and, for n large enough, $F_{q,q_n}(\epsilon) > 1-\mu$. However by Corollary 1.6.7, we have $F_{q_0,q}(\epsilon) > F_{q_0,q_{n+1}}(\epsilon)$ and $F_{q_{n+1},q_n}(\epsilon) > F_{q,q_n}(\epsilon)$ for all n. Since by assumption $F_{q_0,q_{n+1}}(\epsilon) = F_{q_{n+1},q_n}(\epsilon)$ for all n, the earlier inequalities combine give rise $F_{q_0,q}(\epsilon) > F_{q,q_n}(\epsilon)$ for all n, which is a contradiction. This concludes the proof.

Definition 1.6.11 Let $(X, \mathcal{F})$ be a PM-space. Then
(1) a sequence of points $\{p_n\}$ in X is a Cauchy sequence if $F_{p_n \cdot p_m} \to H$ (pointwise) as $n, m \to \infty$, where H is the specific distribution function defined earlier.
(2) the space $(X, \mathcal{F})$ is complete if every Cauchy sequence in X is convergent.
(3) the space $(X, \mathcal{F})$ and $(X', \mathcal{F}')$ are isometric if there is one to one mapping Φ of X onto X' such that for any $p, q \in X$,

$$F(p, q) = F'(\Phi(p), \Phi(q)).$$

where Φ is called an isometry map.
(4) a complete PM-space $(X^*, \mathcal{F}^*)$ is a completion of $(X, \mathcal{F})$ if $(X, \mathcal{F})$ is isometric to a dense subset of $(X^*, \mathcal{F}^*)$.

Definition 1.6.12 Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space. The Menger PM-space $(X^*, \mathcal{F}^*, \Delta^*)$ is a completion of $(X, \mathcal{F}, \Delta)$ if $(X^*, \mathcal{F}^*)$ is a completion of $(X, \mathcal{F})$ and $\Delta = \Delta^*$.

Theorem 1.6.13 Every Menger PM-space with a continuous t-norm has a completion which is unique upto isometry.

Proof: For the proof, one may consult Chang et. al [9].

CHAPTER 2

ON FURTHER THEORY OF PM-SPACES

2.1. Introduction

This chapter is mainly devoted to various properties of PM-spaces with a special emphasis to topological structures, the problem of metrization and the form of metric functions for PM-spaces besides discussing Menger PM-spaces and probabilistic normed linear spaces along with the characterizations of various probabistically bounded sets. As applications, these results are utilized to study the linear operator theory on PM-spaces. In this continuation, the concepts of topology, uniformity, metrics, semi-metrics, random norm and para norm along with functions of non-compactness are discussed. Since 1960 onwards, various aspects of theory and applications of probabilistic metric spaces have been developed by many authors especially Chang et al. $[4-8,10]$, Hadžić $[14,15,17]$, Hicks et al. $[19,21,22]$, Radu $[34-38]$, Schweizer and Sklar $[45,47,48]$, Sherwood $[53-55]$ and others which include results on the topological structures and properties for probabilistic metric spaces, the metrization conditions and the form of metric functions in PM-spaces, Menger PM-spaces and probabilistic normed spaces. The several questions regarding characterizations of various probabilistically bounded sets in the linear operator theory and fixed point theory in PM-spaces remain unanswered. The purpose of this chapter is to investigate relevant results on these related topics.

2.2. Topological structures in PM-space

Let $\mathcal{D}_0$ be the subset of set of all left continuous distribution function $\mathcal{D}$ described by

$$\mathcal{D}_0 = \left\{ f \in \mathcal{D} : f^{-1}(1) \neq \Phi \right\}$$

Theorem 2.2.1. Let $(X, \mathcal{F})$ be a PM-space such that $\mathcal{F}$ takes its values in $\mathcal{D}_0$. If we define a function $d : X \times X \to R^+$ as

$$d(x,y) = \inf_t \left\{ t \geq 0,\ F_{x,y}(t) = 1,\ \forall x, y \in X, \ \right\} \tag{2.2.1.1}$$

then d is a metric on X and so $(X, \mathcal{F})$ is a metric space.

Proof: By the definition of d and conditions PM-1 and PM-3, it is easy to see that

$$d(x,y) \geq 0\,,\; d(x,y) = d(y,x)$$

and

$$d(x,y) = 0 \Longleftrightarrow x = y.$$

By definition of d for any $x, y, z \in X$ and $\epsilon > 0$, we have

$$F_{x,z}(d(x,z) + \frac{\epsilon}{2}) = 1\,,\; F_{z,y}(d(z,y) + \frac{\epsilon}{2}) = 1.$$

Hence from PM-4 , we have

$$F_{x.y}(d(x,z) + d(z,y) + \epsilon) = 1,$$

yielding thereby

$$d(x,y) \leq d(x,z) + d(z,y) + \epsilon.$$

Letting $\epsilon \longrightarrow 0$, we have

$$d(x,y) \leq d(x.z) + d(z,y)$$

Proposition 2.2.2 Let $(X, \mathcal{F})$ be a PM-space where $\mathcal{F}$ take its values in $\mathcal{D}_0$. Let d be the metric function described by (2.2.1.1) and the function d_α , $\alpha \in (0,1]$ is defined by

$$d_\alpha(x,y) = \inf_t \{t \geq 0 : F_{x,y}(t) > 1 - \alpha\}, \qquad x, y \in X \qquad (2.2.2.1)$$

Then for any given $x, y \in X$ and $\alpha \in (0,1]$, d_α is a decreasing function and

$$d(x,y) = \lim_{\alpha \longrightarrow 0} d_\alpha(x,y) = \sup_{\alpha \in (0,1]} d_\alpha(x,y) \qquad (2.2.2.2)$$

Proof: For given $x, y \in X$, if $\alpha_1, \alpha_2 \epsilon (0,1]$ with $\alpha_1 > \alpha_2$, then we have

$$\{t \geq 0,\; F_{x,y}(t) > 1 - \alpha_1\} \supseteq \{t \geq 0,\; F_{x,y}(t) > 1 - \alpha_2\}$$

and so

$$d_{\alpha_1}(x,y) \leq d_{\alpha_2}(x,y)$$

This shows that d_α is a decreasing function in $\alpha \,\epsilon\, (0,1]$ and so the $\lim_{\alpha \longrightarrow 0} d_\alpha(x,y)$ exists and

$$\lim_{\alpha \longrightarrow 0} d_\alpha(x,y) = \sup_{\alpha \epsilon (0,1]} d_\alpha(x,y)$$

By the definition of d, for any $\epsilon > 0$, we have

$$F_{x,y}(d(x,y) + \epsilon) = 1,$$

$\forall\ x, y \in X$, so for any $\alpha \in (0,1]$,

$$\inf_t \{t \geq 0 : \ F_{x,y}(t) > 1-\alpha\} \leq d(x,y) + \epsilon$$

Now,

$$\sup_{\alpha \in (0,1]} d_\alpha(x,y) = \sup_{\alpha \in (0,1]} \inf_t \{t \geq 0,\ F_{x,y}(t) > 1-\alpha\} \leq d(x,y) \tag{2.2.2.3}$$

By definition of d, for any $\epsilon > 0$, $\exists\ \alpha = \alpha_0(\epsilon)$ such that

$$F_{x,y}(d(x,y) - \epsilon) < 1 - \alpha_0.$$

Hence we have

$$\begin{aligned} d(x,y) - \epsilon &\leq \inf_t \{t \geq 0,\ F_{x,y}(t) > 1-\alpha\} \\ &\leq \sup_{\alpha \to (0,1]} \inf_t \{t \geq 0,\ F_{x,y}(t) > 1-\alpha\} \end{aligned}$$

letting $\epsilon > 0$ on the left side

$$d(x,y) \leq \sup_{\alpha \to (0,1]} d_\alpha(x,y) \tag{2.2.2.4}$$

from (2.2.2.3) and (2.2.2.4), we have (2.2.2.2), which completes the proof.

Theorem 2.2.3 Let $(X, \mathcal{F})$ be a PM-space and $\mathcal{F}$ satisfy the following condition : (PM-α): For any $x, y, z \,\epsilon\, X$ and for any $t_1, t_2 > 0$, $\lambda > 0$, if $F_{x,z}(t_1) > 1-\lambda$, $F_{z,y}(t_2) > 1-\lambda$, implies that

$$F_{x,y}(t_1 + \ t_2) > 1 - \lambda.$$

Then
(1) For every $\alpha \in (0,1]$, the function d_α defined by

$$d_\alpha(x,y) = \inf_t \{t \geq 0\text{: } F_{x,y}(t) > 1-\alpha\} \tag{2.2.3.1}$$

is a pseudo metric on X.
(2) For each $\alpha \in (0,1]$, the topology $\mathcal{T}_\alpha$ induced by the family of neighborhoods

$$\{U \ \subset \ X\text{: } \forall\, x \in U, \exists\, \epsilon > 0 : \ N_x(\epsilon, \alpha) \subset U\}$$

where

$$N_x(\epsilon,\alpha) = \{y \ \in \ X\text{: } F_{x,y}(\epsilon) > 1-\alpha\}$$

coincides with the topology $\mathcal{T}_{d_\alpha}$ induced by the family of neighborhoods

$$\{B_\alpha(x,\epsilon) \ : \ x \in \ X,\ \epsilon > 0\} \text{ in } X$$

,where

$$B_\alpha(x,\epsilon) = \{\ y \in \ X : \ d_\alpha(x,y) < \epsilon\}$$

. **Proof:**(1) For each $\alpha \in (0,1]$, it is easy to see that

$$d_\alpha(x,y) \geq 0\ ,\ d_\alpha(x,y) = d_\alpha(y,x)$$

and $d_\alpha(x,y) = 0$ provided $x = y$. For each $\epsilon > 0$, (due to Definition of d_α) we have

$$F_{x,z}(d_\alpha(x,z) + \frac{\epsilon}{2}) > 1-\alpha\ \ ,\ \ F_{z,y}(d_\alpha(z,y) + \frac{\epsilon}{2}) > 1-\alpha.$$

It follows from the definition of (PM-α) that

$$F_{x,y}(d_\alpha(x,z) + d_\alpha(z,y) + \epsilon) > 1-\alpha$$

and so we have

$$d_\alpha(x,y) \leq d_\alpha(x,z) + d_\alpha(z,y) + \epsilon.$$

Letting $\epsilon \longrightarrow 0$, it follows that

$$d_\alpha(x,y) \leq d_\alpha(x,z) + d_\alpha(z,y).$$

Since $d_\alpha(x,y) = 0$ does not imply that $x = y$, for each $\alpha \in (0,1]$,hence d_α is a pseudo metric.

(2) It is sufficient to show that

$$N_x(\epsilon,\alpha) = B_\alpha(x,\epsilon)\,. \tag{2.2.3.2}$$

In fact, if $y \in N_x(\epsilon,\alpha)$, then $F_{x,y}(\epsilon) > 1-\alpha$. By the left continuity of the distribution function, there exists $\epsilon' \in (0,\epsilon)$ such that

$$F_{x,y}\left(\epsilon'\right) > 1-\alpha.$$

and so we have

$$d_\alpha(x,y) = \operatorname*{Inf}_t \{t \geq 0 : F_{x,y}(t) > 1-\alpha\} \leq \epsilon' < \epsilon.$$

which implies that $y \in B_\alpha(x,\epsilon)$, i.e., $N_x(\epsilon,\alpha) \subset B_\alpha(x,\epsilon)$.
Conversely, if $y \in B_\alpha(x,\epsilon) \Rightarrow d_\alpha(x,y) < \epsilon$, therefore

$$\begin{aligned} F_{x,y}(\epsilon) &> 1-\alpha, i.e., y \in N_x(\epsilon,\alpha) \\ B_\alpha(x,\epsilon) &\subset N_x(\epsilon,\alpha) \end{aligned}$$

which completes the proof.

2.3 Topological structures in Menger PM-spaces

B. Schweizer and A.Sklar [45] pointed out that if $(X, \mathcal{F}, \Delta)$ is a Menger PM-space with the t-norm Δ which satisfies

$$\sup_{t<1} \Delta(t,t) = 1,$$

Then $(X, \mathcal{F}, \Delta)$ is a **metrizable Housdorff topological space.**

Theorem 2.3.2 Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space.
(i) If $\mathcal{F}$ takes its values in $\mathcal{D}_0$, then for any t-norm Δ, then $(X, \mathcal{F}, \Delta)$ is a metric space and the metric d is described by (2.2.1.1).
(ii) If t-norm Δ satisfies the following condition

$$\Delta(a,b) \geq \max\{a+b-1, 0\}, \ a, b \in [0,1],$$

then for any $\mathcal{F}$ which satisfies the conditions (PM-1)-(PM-3) and (PM-4), $(X, \mathcal{F}, \Delta)$ is metrizable and the function d^* defined by

$$d^*(x,y) = \sup_t \{t : \ F_{x,y}(t) \leq 1-t\} \tag{2.3.2.1}$$

is a metric on X. In addition, the metric topology on X induced by d^* coincides with the topology on X induced by the family of neighborhoods

$$\{U \subset X : \forall x \in U, \exists \epsilon > 0 \text{ such that } N_x(\epsilon, \epsilon) \subset U\}, \tag{2.3.2.2}$$

where

$$N_x(\epsilon, \epsilon) = \{y \in X : F_{x,y}(t) > 1-\epsilon\}.$$

Proof: The conclusion (i) follows easily from Theorem 2.2.1. Hence, all we require to prove is the condition (ii).
First, we notice that the function d^* defined by (2.3.2.1) has the following properties:

$$d^*(x,y) < t \quad \Leftrightarrow F_{x,y}(t) > 1-t, \ \ t > 0. \tag{2.3.2.3}$$

Now we prove that d^* is a metric on X. Infact, it is obvious that $d^*(x,y) \geq 0$, $d^*(x,y) = d^*(y,x)$ and $d^*(x,y) = 0 \ \Leftrightarrow x = y$. On the other hand by the definition of d^*, for any $\epsilon > 0$ and $x, y, z \in X$, we have

$$\begin{aligned} F_{x,z}\left(d^*(x,z) + \frac{\epsilon}{2}\right) &> 1 - d^*(x,z) - \frac{\epsilon}{2}, \\ F_{z,y}\left(d^*(z,y) + \frac{\epsilon}{2}\right) &> 1 - d^*(z,y) - \frac{\epsilon}{2}. \end{aligned} \tag{2.3.2.4}$$

Hence making use of (PM-4) and the above expressions it follows that

$$\begin{aligned} F_{x,z}\left(d^*(x,z)+d^*(z,y)+\epsilon\right) &\geq \Delta\left(F_{x,z}\left(d^*(x,z)+\frac{\epsilon}{2}\right), F_{z,y}\left(d^*(z,y)+\frac{\epsilon}{2}\right)\right) \\ &\geq F_{x,z}\left(d^*(x,z)+\frac{\epsilon}{2}\right)+F_{z,y}\left(d^*(z,y)+\frac{\epsilon}{2}\right)-1 \\ &> 1-\left(d^*(x,z)+d^*(z,y)+\epsilon\right). \end{aligned}$$

Using (2.3.2.3) , we have

$$d^*(x,y) < d^*(x,z)+d^*(z,y)+\epsilon.$$

Letting $\epsilon \to 0$, we have

$$d^*(x,y) \leq d^*(x,z)+d^*(z,y),\ x,y,z \in X.$$

Lastly, in order to prove that metric topology induced by d^* coincides with the topology $\mathcal{T}$ induced by the family of neighborhoods defined by (2.3.2.2), it suffices to note that using (2.3.2.3) we can prove that

$$N_x(\epsilon,t) = \{y \in X : d^*(x,y) < \epsilon\}.$$

This completes the proof.

Theorem 2.3.3 If $(X,\mathcal{F},\Delta)$ is a Menger PM-space satisfying PM-α .Then the function d^* described by (2.3.2.1) is a metric on X.

Proof: It is sufficient to prove that d^* satisfies the triangle inequality. In fact we know

$$F_{x,z}\ (d^*(x,z)+\frac{\epsilon}{2}) > 1-d^*(x,z)-\frac{\epsilon}{2}$$

and

$$F_{z,y}(d^*(z,y)+\frac{\epsilon}{2}) > 1-d^*(z,y)-\frac{\epsilon}{2}$$

For any $x,y,z \in X$, we have

$$F_{x,z}(d^*(x,z)+\ \frac{\epsilon}{2}) > 1-d^*(x,z)-d^*(z,y)-\epsilon$$

and

$$F_{z,y}(d^*(z,y)+\ \frac{\epsilon}{2}) > 1-d^*(x,z)-d^*(z,y)-\epsilon$$

In view of the condition PM-α, we have

$$F_{x,y}(d^*(x,z)+\ d^*(z,y)+\ \epsilon) > 1-d^*(x,z)-d^*(z,y)-\epsilon.$$

Now from property (2.3.2.3) , one can write

$$\begin{aligned} d^*(x,y) &< t \Longleftrightarrow F_{x,y}(t) > 1-t, \quad t>0 \\ d^*(x,y) &< d^*(x,z) + d^*(z,y) + \epsilon \end{aligned}$$

letting $\epsilon \to 0$, reduces to

$$d^*(x,y) \leq d^*(x,z) + d^*(z,y), \quad \forall x,\ y,\ z\ \in X.$$

This completes the proof.

Remark 2.3.4 If the t-norm $\Delta = \min$, then the mapping $\mathcal{F}$ in a Menger PM-space $(X,\ \mathcal{F},\ \Delta)$ satisfies the condition PM-α. Infact, for any $x, y, z\ \in X$ and $t_1, t_2 > 0$, $\lambda > 0$, if $F_{x,z}(t_1) > 1-\lambda$ and $F_{z,y}(t_2) > 1-\lambda$, then from PM-4, we have

$$F_{x,y}(t_1+t_2) \geq \min\ \{F_{x,z}(t_1)\,,\ F_{z,y}(t_2)\} > 1-\lambda$$

Now by Theorem 2.3.3, $(X, \mathcal{F}, d^*)$ is a metric space.

2.4 Topological structures in PN-spaces

In this section, we discuss the topological structures and the metrization problem in probabilistic normed spaces.

Definition 2.4.1 A probabilistic normed space (PN-space) is an ordered pair $(X, \mathcal{F})$, where X is a real linear space, $\mathcal{F}$ is a mapping from X into $\mathcal{D}$ (We shall denote the distribution function $\mathcal{F}(x)$ by f_x) satisfying the following conditions:
(PN-1) $f_x(t) = 1, \forall t > 0 \Longleftrightarrow x = 0$,
(PN-2) $f_x(0) = 0$,
(PN-3) $f_{\alpha x}(t) = f_x\left(\frac{t}{|\alpha|}\right), \forall \alpha\ \in R,\ \alpha \neq 0$,
(PN-4) if $f_x(t_1) = 1$ and $f_y(t_2) = 1$, then $f_{x+y}(t_1+t_2) = 1$.
By definition, it is obvious that if we take $F_{x,y} = f_{x-y}$, then PN-space must be a PM-space.

Definition 2.4.2 A Menger PN-space is a triple $(X, \mathcal{F}, \Delta)$, where $(X, \mathcal{F})$ is a PN-space and Δ is a t- norm satisfying the following condition:
(PN-4$_m$) $f_{x+y}(t_1+t_2) \geq \Delta(f_x(t_1)\,,\ f_y(t_2)), \forall x, y\ \in X,\ t_1, t_2\ \in R^+$.

Theorem 2.4.3 Let $(X, \mathcal{F})$ be a PN-space and $\mathcal{F}$ takes its values in $\mathcal{D}_0$. If we define the function $\|.\|$ and $\|.\|_\alpha$, $\alpha \in (0,1]$, as follows respectively,

$$\begin{aligned} \|x\| &= \inf_t \{t \geq 0,\ f_x(t) = 1\} \qquad (2.4.3.1) \\ \|x\|_\alpha &= \inf_t \{t \geq 0,\ f_\alpha(t) > 1-\alpha\}, \end{aligned}$$

$\forall\, x \in X$ and $\alpha \in (0,1]$, then
(*i*) $\|.\|$ is a norm on X and so $(X, \mathcal{F}, \|.\|)$ is a normed linear space.
(*ii*) $\|.\|_\alpha$ is monotone decreasing in $\alpha \in (0,1]$ and

$$\|x\| = \lim_{\alpha} \|x\|_\alpha = \sup_{\alpha \in (0,1]} \|x\|_\alpha \tag{2.4.3.2}$$

Proof: (i) It is sufficient to show that $\|.\|$ satisfies the norm conditions:
(*a*) From PN-1 & PN-2 , it is easy to see that $\|x\| \geq 0$ & $\|x\| = 0 \iff x = 0$.

(*b*) From PN-3, for any $\alpha \in R$, $\alpha \neq 0$, we have

$$\begin{aligned} \|\alpha x\| &= \inf_t \{t \geq 0 : f_{\alpha x}(t) = 1\} \\ &= \inf_t \left\{t \geq 0 : f_x\left(\frac{t}{|\alpha|}\right) = 1\right\} \\ &= |\alpha| \inf_t \{t \geq 0 : f_x(t) = 1\} = |\alpha|\, \|x\| \end{aligned}$$

If $\alpha = 0$, the above inequality still holds.

(*c*) By definition of $\|.\|$, for any $\in > 0$, we have

$$f_x\left(\|x\| + \frac{\epsilon}{2}\right) = 1,\ f_y\left(\|y\| + \frac{\epsilon}{2}\right) = 1, \quad \forall\, x, y\ \in X$$

Hence from PN-4, it follows that

$$f_{x+y}\left(\|x\| + \|y\| + \epsilon\right) = 1, \forall\, x, y\ \in X$$

and so

$$\|x + y\| \leq \|x\| + \|y\| + \epsilon$$

letting $\epsilon \to 0$, we have

$$\|x + y\| \leq \|x\| + \|y\|\,.$$

which shows that $\|.\|$ is a norm on X.

(*ii*) it can be shown on the lines of Proposition 2.2.2.

Remark 2.4.4 Let $(X, \mathcal{F})$ be a PN-space and $\mathcal{F}$ satisfies the condition PN-β : For any $x, y \in X$ and $t_1, t_2 > 0$, $\lambda > 0$, if

$$f_x(t_1) > 1 - \lambda \text{ and } f_y(t_2) > 1 - \lambda$$

then

$$f_{x+y}(t_1 + t_2) > 1 - \lambda$$

2.5 Further topological properties of PM-spaces

In this section, we introduce the topological concepts in a metric spaces.

Definition 2.5.1 Let $(X,\mathcal{F})$ be a PM-space, where $\mathcal{F}$ takes its values in $\mathcal{D}_0$.
(i) A sequence $\{x_n\}$ in X is said to be d-convergent to $x \in X$, where d is defined by (2.2.1.1) , denote it by $x_n \xrightarrow{d} x$,
if for $\epsilon > 0, \exists\ N = N(\epsilon)$ such that $F_{x_n,x}(\epsilon) = 1$ whenever $n \geq N$.

(ii) A sequence $\{x_n\}$ in X is d-Cauchy sequence if for any $\epsilon > 0$, $\exists\ N = N(\epsilon)$, such that $F_{x_m,x_n}(\epsilon) = 1$, whenever $n, m \geq N$.

(iii) $(X,\mathcal{F})$ is said to be d-complete if every d-Cauchy sequence in X is d- convergent to some point in X.

(iv) A set $A \subset (X,\mathcal{F})$ is said to be d-ϵ -dense in X if for each $x \in X$ and $\epsilon > 0$, $\exists$ an $x^* \in A$ such that

$$F_{x,x^*}(\epsilon) = 1.$$

(v) $(X,\mathcal{F})$ is said to be d-totally bounded if for each $\epsilon > 0$, $\exists$ a finite subset $A \subset X$ such that A is d-ϵ- dense in X.

Definition 2.5.2 Let $(X,\mathcal{F},\Delta)$ be a Menger PM-space and $\mathcal{F}$ satisfy the condition PM-5 or Δ satisfy the condition :

$$\Delta(a,b) \geq max\{a+b-1, 0\}, \quad a, b \in [0,1].$$

Suppose that d^* is defined by (2.3.2.1)

(i) A sequence $\{x_n\}$ in X is said to be d^*- convergent to $x \in X$, $x_n \xrightarrow{d^*} x$, if for any given $\epsilon > 0$, $\exists$ a positive integer $N = N(\epsilon)$ such that $F_{x_n,x}(\epsilon) > 1-\epsilon$, whenever $n \geq N$.

(ii) A sequence $\{x_n\}$ in X is said to be d^*-Cauchy sequence if $\forall\ \epsilon > 0$, $\exists\ N = N(\epsilon)$ such that

$$F_{x_n,x_m}(\epsilon) > 1-\epsilon, \text{ whenever } \ n, m \geq N(\epsilon).$$

(iii) $(X,\mathcal{F}.\Delta)$ is said to be d^* -complete if every d^*-Cauchy sequence in X is d^*-convergent to some point in X.

(iv) A subset $A \subset X$ is said to be d^*-ϵ-dense if $\forall\ x \in X$, $\exists$ an $x^* \in A$, such that

$$F_{x_*,x}(\epsilon) > 1-\epsilon.$$

(*v*) $(X, \mathcal{F}, \Delta)$ is said to be d^*-totally bounded if $\forall\ \epsilon > 0$, there exists a finite subset A such that A is d^*-ϵ-dense in X.

Theorem 2.5.3 Let $(X, \mathcal{F})$ be a PM-space and $\mathcal{F}$ take its values in $\mathcal{D}_0$. Then X is d-complete, (where d is defined by (2.2.1.1)) if and only if for each family $\mathcal{L}$ of d-closed subsets of X satisfying the following conditions:
(*i*) $\mathcal{L}$ has the finite intersection property,
(*ii*) For each $\epsilon > 0$, there exists $A \in \mathcal{L}$ such that $D_A(\epsilon) = 1$, where

$$D_A(t) = \sup_{s<y} \inf_{x,y \in A} F_{x,y}(s), \quad t \in R,$$

$$\cap_{M \in \mathcal{L}} M \neq \Phi.$$

Proof: Due to the limitation of the space, we do not include the proof of this theorem. For the proof, one may consult Chang et. al [9].

2.6 Probabilistically bounded sets in PM-spaces

Definition 2.6.1 Let A be a non empty subset of a PM-space $(X, \mathcal{F})$. Then
(*i*) A is said to be probabilistically uniformly bounded if $\exists\ M > 0$ such that $F_{x,y}(M) = 1, \forall x, y \in A$,

(*ii*) A is said to be probabilistically semi bounded if $0 < \sup_{t>0} \inf_{x,y \in A} F_{x,y}(t) < 1$,

(*iii*) A is said to be probabilistically bounded if $\sup_{t>0} \inf_{x,y \in A} F_{x,y}(t) = 1$,

(*iv*) A is said to be probabilistically unbounded if $\sup_{t>0} \inf_{x,y \in A} F_{x,y}(t) = 0$.

(*iv*) If X (as a subset of X) itself is Probabilistically uniformly bounded (respectively semi bounded, bounded, unbounded), then X is said to be probabilistically uniformly bounded (respectively semi bounded, bounded, unbounded) space.

Theorem 2.6.2 Let $(X, \mathcal{F})$ be a PM-space and A be a non empty subset of X. If A is probabilistically uniformly bounded then $\exists$ an $M > 0$ such that the following inequalities uniformly holds with respect to $\alpha \in (0, 1]$,

$$\sup_{x,y \in A} d_\alpha(x, y) \leq M, \qquad (2.6.2.1)$$

where

$$d_\alpha(x, y) = \inf_t \{t \geq 0 : F_{x,y}(t) > 1 - \alpha\} \qquad (2.6.2.2)$$

Proof: Suppose that A is probabilistically uniformly bounded, then by definition $\exists$ an $M > 0$ such that

$$F_{x,y}(M) = 1, \forall x, y \in A.$$

Hence, for all $\alpha \in (0,1]$, we have

$$F_{x,y}(M) > 1-\alpha, \quad x, y \in A.$$

so that

$$\inf_t \{t \geq 0 : F_{x,y}(t) > 1-\alpha\} \leq M,$$

which implies that the following inequality uniformly holds with respect to $\alpha \in (0,1]$,

$$\sup_{x,y \in A} d_\alpha(x,y) \leq M.$$

This completes the proof of the theorem.

Theorem 2.6.3 Let A be a subset of a PM-space $(X, \mathcal{F})$. Then A is probabilistically bounded iff $\exists G \in \mathcal{D}$ with $G(0) = 0$, such that $\forall\, x, y \in A$ and $t \geq 0$,

$$F_{x,y}(t) \geq G(t). \tag{2.6.3.1}$$

Proof: If A is probabilistically bounded set,

$$\sup_{t>0} \sup_{s<t} \inf_{x,y \in A} F_{x,y}(s) = \sup_{t>0} \inf_{x,y \in A} F_{x,y}(t) = 1.$$

Take

$$G(t) = \sup_{s<t} \inf_{x,y \in A} F_{x,y}(s).$$

It is easy to prove that $G \in \mathcal{D}$, $G(0) = 0$ and $\forall\, x, y \in A$ and $t \geq 0$,

$$G(t) \leq \inf_{x,y \in A} F_{x,y}(t) \leq F_{x,y}(t).$$

Conversely, if $\exists$ a $G \in \mathcal{D}$, $G(0) = 0$, such that (2.6.3.1) holds, then we have

$$\inf_{x,y \in A} F_{x,y}(t) \geq G(t)$$

and so,

$$\sup_{t>0} \inf_{x,y \in A} F_{x,y}(t) \geq \sup_{t>0} G(t) = 1.$$

This shows that A is probabilistically bounded.

2.7 Linear operators in PN-spaces

Definition 2.7.1 Let $(X, \mathcal{F})$ be a PN-space and T be a linear operator from X into X.
(i) T is said to be strongly bounded If $\exists$ an $M > 0$ such that

$$f_{Tx}(t) \geq f_{Mx}(t), \quad \forall x \in X, \; t > 0, \tag{2.7.1.1}$$

where $f_y = \mathcal{F}(y)$, $y \in X$.

(ii) T is said to be bounded if T maps each probabilistically bounded set into a probabilistically bounded set.
We have already shown that if $(X, \mathcal{F})$ is a PN-space and $\mathcal{F}$ takes its values in $\mathcal{D}_0$, Then $(X, \mathcal{F}, \|.\|)$ is a linear normed space, where $\|.\|$ defined by

$$\|x\| = \inf_t \{t \geq 0 : \; f(x) = 1\}, \quad x \in X. \tag{2.7.1.2}$$

In this case, a sequence $\{x_n\} \subseteq X$ is said to be convergent in norm $\|.\|$ to $x \in X$, if for any $\epsilon > 0$, $\exists$ a positive integer $N = N(\epsilon)$ such that

$$\begin{aligned} \|x_n - x\| &= \inf_t \{t \geq 0 : f_{x_n - x}(t) = 1\} < \epsilon, \text{ whenever } n \geq N, \\ i.e., \; f_{x_n - x}(\epsilon) &= 1, \text{ whenever } n \geq N(\epsilon) \end{aligned}$$

.Naturally, a mapping $T : X \longrightarrow X$ is said to be continuous in $\|.\|$ if for any sequence $\{x_n\}$ in X which converges in norm $\|.\|$ to x, implies that

$$\|T_{x_n} - T_x\| \longrightarrow 0,$$

whereas a mapping $T : X \longrightarrow X$ is said to be $\mathcal{T}$- continuous if for any sequence $\{x_n\}$ in X such that

$$x_n \xrightarrow{\mathcal{T}} x, \text{ implies that } T_{x_n} \xrightarrow{\mathcal{T}} T_x.$$

Lemma 2.7.2 Let $(X, \mathcal{F})$ be a PN-space and $\mathcal{F}$ takes its value in $\mathcal{D}_0$. Let $\|.\|$ be norm on X defined by (2.7.1.2) Suppose $A \subseteq X$. Then A is probabilistically uniformly bounded $\Longleftrightarrow \exists \; M > 0$ such that $\sup_{x \in A} \|x\| < M$.

Proof: If A is probabilistically uniformly bounded, then $\exists \; M' > 0$ such that the following inequality holds with respect to $\alpha \in (0, 1]$;

$$M'' \triangleq \sup_{x, y \in A} \|x - y\|_\alpha < M', \tag{2.7.2.1}$$

where

$$\|x\|_\alpha = \inf_t \{t \geq 0 : f_x(t) > 1 - \alpha\}.$$

Hence, for any $x, y \in A$ and $\alpha \in (01]$,

$$\|x - y\|_\alpha \leq M''.$$

Thus letting $\alpha \to 0$, we have

$$\sup_{x,y \in A} \|x - y\| \leq M''. \tag{2.7.2.2}$$

Let $y_0 \in A$ be any fixed point. Then

$$\sup_{x \in A} \|x\| \leq \sup_{x \varepsilon A} \|x - y_0\| + \|y_0\| \leq M'' + \|y_0\| < M' + \|y_0\| \triangleq M.$$

Conversely, if $\exists\ M > 0$ such that

$$\sup_{x \varepsilon A} \|x\| < M.$$

It follows that

$$\sup_{x,y \in A} \|x - y\|_\alpha \leq \sup_{x,y \in A} \|x - y\| < 2M,$$

which implies that A is probabilistically uniformly bounded.

Theorem 2.7.3 Let $(X, \mathcal{F})$ be a PN-space with $\mathcal{F}$ satisfying (PN-β). Then every strongly bounded linear operator $T : X \longrightarrow X$ is bounded.

Proof: Since $T : X \longrightarrow X$ is strongly bounded linear operator, therefore by definition $\exists$ an $M > 0$ such that, $\forall\ x \in X$ and $t \geq 0$,

$$f_{Tx}(t) \geq f_{Mx}(t).$$

Hence we have

$$\inf_t \{t \geq 0\colon\ f_{Tx}(t) > 1 - \alpha\} \leq \inf_t \{t \geq 0\colon\ f_{Mx}(t) > 1 - \alpha\}, \quad \alpha \in (0, 1],$$

($\forall\ x \varepsilon\ X$ and $\alpha \in (0, 1]$), which implies that

$$P_\alpha(Tx) \leq P_\alpha(Mx) = M.(P_\alpha(x)) \tag{2.7.3.1}$$

If $A \subseteq X$ is probabilistically bounded, then $\forall\ \alpha \in (0, 1]$, $\exists\ M'(\alpha) > 0$ such that

$$\sup_{x \varepsilon A} P_\alpha(x) < M'(\alpha) \tag{2.7.3.2}$$

Using (2.7.3.1) and (2.7.3.2), we have

$$\sup_{x \varepsilon A} P_\alpha(Tx) \leq M. \sup_{x \varepsilon A} P_\alpha(x) < M.M'(\alpha).$$

2.8 Topology, uniformity, metrices and semi-metrices on PM-spaces

As expected, various kinds of topologies can be introduced in a PM-spaces see [47]. The strong topology is introduced by a strong neighborhood system $N = \bigcup_{p \epsilon S} N_p$, where

$$N_p = \{N_p(t) \mid t > 0\}$$

and

$$N_p(t) = \{q \mid F_{p,q}(t) > 1 - t\}$$

for $t > 0$ and $p \in S$.

Theorem 2.8.1 If $(X, \mathcal{F}, \Delta)$ be a PM-spaces with Δ continuous, then the strong neighborhood system N determines a Housdorff topology on S.

It is our special interest the (ϵ, λ)-topology on $(X, \mathcal{F}, \Delta)$ which is introduced by a family of neigbourhoods

$$(N_p(\epsilon, \lambda))_{p \in S, \epsilon > 0, \lambda \in (0,1)}$$

.where

$$N_p(\epsilon, \lambda) = \{q \mid q \in S, F_{p,q}(\epsilon) > 1 - \lambda\}.$$

Since

$$N_p(t,t) = N_p(t) \quad \text{for } t > 0$$

and

$$N_p(\min\ (\epsilon, \lambda)) \subseteq N_p\ (\epsilon, \lambda) \quad \forall \epsilon > 0, \lambda \in (0,1)$$

the strong neighborhood system is equivalent to the (ϵ, λ)-neighborhood system.

Remark 2.8.2 If $(X, \mathcal{F}, T)$ is a Menger space and $\sup_{a<1} T(a,a) = 1$, then the family $(\mathcal{N}_p)_{p \in X}$ defines on X a metrizable topology. A sequence $(p_n)_{n \in N}$ in X converges in the (ϵ, λ)-topology to $p \epsilon X$ if $\forall\ \epsilon > 0$ and $\lambda \in (0,1)\ \exists\ n_0(\epsilon, \lambda) \in N$ such that

$$F_{p_n,p}(\epsilon) > 1 - \lambda\ \forall n \geq n_0(\epsilon, \lambda).$$

Remark 2.8.3 It is known that if $(X, \mathcal{F}, T)$ is a Menger space, where $\sup_{a<1} T(a,a) = 1$, then the family $\{N(\epsilon, \lambda) \mid \epsilon > 0, \lambda \in (0,1)\}$ is a base for the Housdorff uniformly u_F in X, where

$$N(\epsilon, \lambda) = \{(p,q) \in X \times X,\ F_{p,q}(\epsilon) > 1 - \lambda\}.$$

Remark 2.8.4 A sequence $(x_n)_{n \in N}$ in X is a Cauchy sequence if for every $\epsilon > 0$ and $\lambda \in (0,1)$ there exists $n_0(\epsilon, \lambda) \in N$ such that

$$F_{x_n,x_m}(\epsilon) > 1 - \lambda,\ \forall n, m \geq n_0(\epsilon, \lambda),$$

and X is complete if every Cauchy sequence in X converges to a point in X.

Remark 2.8.5 It was J.F.C. Kingman who firstly constructed a deterministic metric for a Wald space, which is a Menger space $(X, F, \mathcal{T}_{T_P})$. If $(X, F, *)$ is any Wald space

$$d(p,q) = -\log \int_0^\infty e^{-x}\, d\, F_{p,q}(x)$$

is a metric on S. The uniformity derived from the metric d, is equivalent to the uniformity $\mathcal{U}_F$.

Remark 2.8.6 G.L. Cain and R. Kasriel introduced the (ϵ, λ)-topology in $(X, \mathcal{F}, \mathcal{T}_{T_M})$ by a family of pseudo-metric $(d_\alpha)_{\alpha\epsilon(0,1)}$, where

$$d_\alpha(p,q) = \sup\{x \mid x \in R,\ F_{p,q}(x) \leq 1-\alpha\} \ (p,q \in S).$$

The problem of metrization of probabilistic metric space is investigated in [47, 63] and a family of deterministic metrics on a Menger space is given by Radu [40] in the following way.
Let $\mathcal{M}$ be the family of all mappings $m : \bar{R} \to \bar{R}$ $\left(\bar{R} = [0,\infty]\right)$ such that the following conditions are satisfied:
(a) $\forall\, t, s \geq 0 :\ m(t+s) \geq m(t) + m(s)$;
(b) $m(t) = 0 \Leftrightarrow t = 0$;
(c) m is continuous.

Let $(X, \mathcal{F}, \mathcal{T}_{T_P})$ be a Menger space with an Archimedian t-norm T which has an additive generator f . If $m_1, m_2 \in \mathcal{M}$, then the metric d_{m_1,m_2}, given by

$$d_{m_1,m_2}(p,q) = \sup\{t \mid t \geq 0, m_1(t) \leq f \circ F_{p,q}(m_2(t))\} \quad (p,q \in X)$$

defines the (ϵ, λ)-topology on X.
In a special case when $T = T_L$, then $d_{m_1,m_2} : X \times X \to [0,\infty)$ $(m_1, m_2 \in \mathcal{M})$ is defined by

$$d_{m_1,m_2}(p,q) = \sup\{s \mid s \geq 0, m_1(s) \leq 1 - F_{p,q}(m_2(s))\} \quad (p,q \in X).$$

if $m_1(s) = m_2(s) = s$, $\forall\, s \in R$, we obtain the following metric

$$d(p,q) = \sup\{s \mid s \in R, s \leq 1 - F_{p,q}(s)\} \ (p,q \in X).$$

Theorem 2.8.7 Let $(b_n)_{n\epsilon N}$ be an increasing sequence from the interval $[0,1)$ and $\lim\limits_{n\to\infty} b_n = 1$. If $(X, \mathcal{F}, T)$ is a Menger space such that $T(b_n, b_n) = b_n$ for every $n \in N$, then $r_n : X \times X \to [0,\infty)$ $(n \in N)$ defined by

$$r_n(x,y) = \inf\{t \mid t > 0,\ F_{x,y}(t) \geq b_n\}$$

is a pseudo-metric on S and if the uniformity $\mathcal{U}_1$ is equal to the uniformity $\mathcal{U}_R$, given by the family

$$R = (r_n)_{n \in N},$$

then

$$\mathcal{U}_1 = \mathcal{U}_F.$$

Proof: Since $F_{x,x}(t) = 1$, $\forall\, t > 0$, we have that

$$r_n(x,x) = \inf \{t \mid t > 0,\ F_{x,x}(t) \geq b_n\} = \{t \mid t > 0\} = 0.$$

Suppose that $r_n(x,y) = 0$, $\forall\, n \in N$. Since the condition

$$F_{x,y}(t) \geq b_n, \quad \forall t > 0,$$

is equivalent with

$$r_n(x,y) = 0,$$

we obtain by $\lim\limits_{n\to\infty} b_n = 1$ that $F_{x,y}(+0) = 1$ and therefore $x = y$. It is obvious that r_n is symmetric. It remains to prove that r_n satisfies the triangle inequality $\forall\, n \in N$. Let $x, y, z \in S$. In order to prove that $\forall\, n \in N$

$$r_n(x,y) \leq r_n(x,z) + r_n(z,y). \tag{2.8.9.1}$$

We suppose that $r_n(x,z) < a$ and $r_n(z,y) < b$. Then

$$F_{x,z}(a) \geq b_n,\ F_{x,y}(b) \geq b_n$$

and therefore

$$F_{x,z}(a+b) \geq T(F_{x,z}(a), F_{z,y}(b)) \geq T(b_n, b_n) = b_n.$$

We suppose that $r_n(x,z) \leq a + b$ and (2.8.9.1) is proved. We shall prove that $\mathcal{U}_1 = \mathcal{U}_F$, i.e., that the family of pseudo-metrics $(r_n)_{n \in N}$ induces the uniformity $\mathcal{U}_F$. It is obvious that $r_n \leq r_{n+1}$, $\forall\, n \in \mathbb{N}$, and that for every $n \in \mathbb{N}$ we have that $r_n(x,y) < a$ implies $F_{x,y}(a) \geq b_n$. On the other hand

$$F_{x,y}(\epsilon) > 1 - \lambda,\ 1 - \lambda \geq b_n \Rightarrow r_n(x,y) < \epsilon.$$

CHAPTER-3

TRIANGULAR NORMS

3.1 Introduction

Triangular norms first appeared (in the framework of PM-spaces) in the work Menger [28] (also see [26, 47]). By now, it turns out to be a crucial operation in several fields which include fuzzy sets, fuzzy logics (see [26]), theory of generalized measures , non-linear differential and difference equations [32].

In this chapter we intend to visit basic results from the theory of triangular norms (see [26]) which also include some new results closely related to the theory of the fixed point in PM-spaces. Section 3.2 is devoted to basic definitions and some important examples of triangular norms and co-norms. In Section 3.2, we give some basic properties of triangular norms. The ordinal sum construction of t-norms is discussed in Section 3.3 whereas Section 3.4 is devoted to the representation of continuous t-norms. Some important classes of t-norms with left continuous diagonals in the point $(1,1)$ and t-norms of H-type are discussed in Section 3.6 and Section 3.7, respectively. Section 3.8 is devoted to the practical conditions for comparison of t-norms with respect to the pointwise order as well as the domination relation between t-norms, especially important for the product of PM-spaces. Countable infinitary comparison of t-norms with respect to special classes of t-norms is investigated in the last section.

3.2 Triangular norms and conorms

Definition 3.2.1 A triangular norm (in short a t-norm) is a binary operation on the unit interval $[0,1]$, i.e. a function T: $[0,1]^2 \longrightarrow [0,1]$ such that $\forall\ x,y,z \in [0,1]$, the following four axioms are satisfied:

(T-1) $T(x,y) = T(y,x)$ (Commutativity);
(T-2) $T(x,T(y,z)) = T(T(x,y),z)$ (Associativity)
(T-3) $T(x,y) \leq T(x,z)$ whenever $y \leq z$ (Monotonicity),
i.e., this is monotonic non decreasing in the second argument.
(T-4) $T(x,1) = x$ (Boundary condition).

The commutativity (T-1), the monotonicity (T-3), and the boundary condition (T-4) imply that $\forall$ t-norm T and $\forall\ x \in [0,1]$. The following boundary conditions are also satisfied:

$$T(x,1) = T(1,x) = x,$$

$$T(x,0) = T(0,x) = 0$$

and therefore all t-norms coincide on the boundary of the unit square $[0,1]^2$.
The monotonicity of a t-norm T in its second component (T-3), together with the commutativity (T-1), is equivalent to the (joint) monotonicity in both components, i.e., to

$$T(x_1, y_1) \leq T(x_2, y_2) \qquad \text{whenever } x_1 \leq x_2 \text{ and } y_1 \leq y_2. \tag{3.2.1.1}$$

Definition 3.2.2 If T is a t-norm, then its dual t-conorm S: $[0,1]^2 \longrightarrow [0,1]$ is given by

$$S(x,y) = 1 - T(1-x, 1-y).$$

It is obvious that a t-conorm is a commutative, associative and monotone operation on $[0,1]$ with unit element 0. In order to verify associative property, write

$$\begin{aligned}
S(x, S(y,z)) &= 1 - T(1-x, 1 - S(y,z)) \\
&= 1 - T(1-x, 1 - (1 - T(1-y, 1-z))) \\
&= 1 - T(1-x, T(1-y, 1-z)) \\
&= 1 - T(T(1-x, 1-y), 1-z) \\
&= S(S(x,y), z) \\
S(x, S(y,z)) &= S(S(x,y), z)
\end{aligned}$$

which shows that S is associative.

Example 3.2.3 The following are the four basic t-norms together with their dual t-conorms:
(i) Minimum T_M and maximum S_M given by

$$\begin{aligned}
T_M(x,y) &= \min(x,y), \\
S_M(x,y) &= \max(x,y).
\end{aligned}$$

(ii) Product T_P and probabilistic sum S_P given by

$$\begin{aligned}
T_P(x,y) &= x.y, \\
S_P(x,y) &= x + y - x.y.
\end{aligned}$$

(iii) Lukasiewicz t-norm T_L and Lukasiewicz t-conorm S_L given by

$$\begin{aligned}
T_L(x,y) &= max(x + y - 1, 0), \\
S_L(x,y) &= min(x + y, 1).
\end{aligned}$$

(iv) Weakest t-norm (drastic product) T_D and strongest t-conorm S_D given by

$$T_D(x,y) = \begin{Bmatrix} \min(x,y), & \text{if } \max(x,y) = 1, \\ 0, & \text{otherwise,} \end{Bmatrix}$$

$$S_D(x,y) = \begin{Bmatrix} \max(x,y), & \text{if } \min(x,y) = 0, \\ 1, & \text{otherwise} \end{Bmatrix}.$$

Example 3.2.4 (i) The family $\left(T_\lambda^F\right)_{\lambda \in [0,\infty]}$ of Frank t-norms is given by

$$T_\lambda^F(x,y) = \begin{Bmatrix} T_M(x,y) & \text{if } \lambda = 0, \\ T_P(x,y) & \text{if } \lambda = 1, \\ T_L(x,y) & \text{if } \lambda = \infty, \\ \log_\lambda\left(1 + \frac{(\lambda^x-1)(\lambda^y-1)}{\lambda-1}\right) & \text{otherwise.} \end{Bmatrix}.$$

The family $\left(S_\lambda^F\right)_{\lambda \in [0,\infty]}$ of Frank t-conorms is given by

$$S_\lambda^F(x,y) = \begin{Bmatrix} S_M(x,y) & \text{if } \lambda = 0, \\ S_P(x,y) & \text{if } \lambda = 1, \\ S_L(x,y) & \text{if } \lambda = \infty, \\ 1 - \log_\lambda\left(1 + \frac{(\lambda^{1-x}-1)(\lambda^{1-y}-1)}{\lambda-1}\right) & \text{otherwise} \end{Bmatrix}. \quad (3.2.4.1)$$

(ii) The family $\left(T_\lambda^Y\right)_{\lambda \in [0,\infty]}$ of Yager t-norms is given by

$$T_\lambda^Y(x,y) = \begin{Bmatrix} S_D(x,y) & \text{if } \lambda = 0, \\ S_M(x,y) & \text{if } \lambda = \infty, \\ min\left(1, \left(x^\lambda + y^\lambda\right)^{\frac{1}{\lambda}}\right) & \text{otherwise.} \end{Bmatrix}. \quad (3.2.4.2)$$

(iii) Another interesting t-norm is the nilpotent minimum T^{nM} given by

$$T^{nM}(x,y) = \begin{Bmatrix} min(x,y) & \text{if } x + y > 1, \\ 0 & \text{otherwise.} \end{Bmatrix}.$$

If, for two t-norms T_1 and T_2, the inequality $T_1(x,y) \leq T_2(x,y)$ holds $\forall\ (x,y) \in [0,1]^2$, then we say T_1 is weaker than T_2 or, equivalently, that T_2 is stronger than T_1. We shall write $T_1 < T_2$ whenever $T_1 \leq T_2$ and $T_1 \neq T_2$, *i.e.*, if $T_1 \leq T_2$, but $T_1(x_0,y_0) < T_2(x_0,y_0)$ holds for some $(x_0,y_0) \in [0,1]^2$.

Remark 3.2.5 In respect of a t-norm T, an element $x \in [0,1]$ with $T(x,x) = x$ is called an idempotent element of T. It is immediate that 0 and 1 are idempotent elements (which are termed as trivial idempotent elements) for every t-norm.

Remark 3.2.6 The set of idempotent elements is equal to $[0,1]$ in case of minimum T_M, and $\{0\} \cup (0.5, 1]$ in case of the nilpotent minimum T^{nM}; all the other t-norms mentioned thus far have only trivial idempotents.

Proposition 3.2.7 (*i*) The minimum T_M is the only t-norm satisfying $T(x,x) = x \ \forall x \in (0,1)$ (*i.e.*, $\forall x \in [0,1]$ is an idempotent element).
(*ii*) The weakest t-norm T_D is the only t-norm satisfying $T(x,x) = 0 \ \forall x \in (0,1)$.

Proof: If for a t-norm T we assume $T(x,x) = x$, $\forall x \in (0,1)$, then for $y \leq x < 1$, monotonicity (T-3) implies

$$y = T(y,y) \leq T(x,y) \leq min\ (x,y) = y.$$

Together with (T-1) and the boundary conditions this gives exactly $T = T_M$.
If we assume $T(x,x) = 0$, $\forall x \in (0,1)$, then we obtain for each $y \in [0,x)$,

$$0 \leq T(x,y) \leq T(x,x) = 0,$$

yielding thereby $T = T_D$.

3.3 Properties of t-norms

It is well known but interesting to recall that a real function of two variables with domain $[0,1]^2$ can be continuous in each variable without being continuous on $[0,1]^2$. But triangular norms are exceptions to this.

Proposition 3.3.1 A t-norm T is continuous if and only if it is continuous in its first component (i.e., if $\forall y \in [0,1]$ the one place function $T(.,y) = [0,1] \longrightarrow [0,1]$, $x \mapsto T(x,y)$, is continuous).

Proof: If a function from $[0,1]^2$ into $[0,1]$ is continuous, then it is obviously continuous in each component.

Conversely, if T is continuous in the first component, then fix $(x_0, y_0) \in [0,1]^2$ with $\epsilon > 0$ and let $(x_n)_{n\epsilon N}$ and $(y_n)_{n\epsilon N}$ be sequences in $[0,1]$ converging to x_0 and y_0 respectively. From this, we can construct four monotone sequences $(a_n)_{n\epsilon N}$, $(b_n)_{n\epsilon N}$, $(c_n)_{n\epsilon N}$ and $(d_n)_{n\epsilon N}$ such that $\forall n \in N$, we have

$$\begin{aligned} a_n &\leq x_n \leq b_n \text{ and } (a_n)_{n\epsilon N} \nearrow x_0,\ (b_n)_{n\epsilon N} \searrow x_0, \\ c_n &\leq y_n \leq d_n \text{ and } (c_n)_{n\epsilon N} \nearrow y_0,\ (d_n)_{n\epsilon N} \searrow y_0. \end{aligned}$$

The continuity in the first component and the commutativity of T imply the continuity of the function $T(x_0, .)$, which means that there exists an $N \in \mathbb{N}$ such that, as a consequence of the monotonicity of T, for all $n \geq N$,

$$T(x_0, y_0) - \epsilon < T(x_0, c_N) \leq T(x_0, y_n) \leq T(x_0, d_N) < T(x_0 + y_0) + \epsilon.$$

Since also the two functions $T(., C_N)$ and $T(., d_N)$ are continuous there is a number $M \in N$ such that $\forall\, m \geq M$ and $n \geq N$ (again taking into account the monotonicity of T) we obtain

$$\begin{aligned} T(x_0, C_N) - \epsilon &< T(a_M, C_N) \\ &\leq T(x_m, y_n) \\ &\leq T(b_M, d_N) \\ &< T(x_0, d_N) + \epsilon \end{aligned}$$

putting $k = \max\ (M, N)$, then for all $k \geq K$, we have

$$T(x_0, y_o) - 2\epsilon < T(x_k, y_k) < T(x_0, y_0) + 2\epsilon,$$

proving that $(T(x_k, y_k))_{k\epsilon N}$ converges to $T(x_0, y_0)$, i.e., the t-norm T is continuous in (x_0, y_0).

Definition 3.3.2 (i) A t-norm T is said to be strictly monotone if it is strictly increasing on $[0,1]^2$ as a function from $[0,1]^2$ into $[0,1]$, or equivalently in vie of the commutativity (T-1) and the boundary conditions (T-4) .

$$T(x, y) < T(x, z) \quad \text{whenever } x\epsilon\ (0,1) \text{ and } y < z.$$

(ii) A t-norm T is called strict if it is continuous and strictly monotone.

Among the some basic t-norms presented in Example 3.2.3 only the product T_P is a strict t-norm. The minimum T_M and Lukasiewicz t-norm T_L are continuous but not strictly monotone.

Example 3.3.3 The t-norm T defined by

$$T^{\Delta}(x, y) = \begin{cases} \frac{xy}{2} & \text{if } \max\ (x, y) < 1, \\ xy & \text{othewise.} \end{cases} \tag{3.3.3.1}$$

is strictly monotone but not continuous.

Example 3.3.4 If T is a strictly monotone t-norm , $x \in [0,1]$ and $n \in N \cup \{0\}$, then we shall write

$$x_T^{(n)} = \left\{ \begin{array}{ll} 1 & \text{if } n = 0 \\ T\left(x_T^{(n-1)}, x\right) & \text{otherwise} \end{array} \right\}. \tag{3.3.4.1}$$

If T is a strictly monotone t-norm then $\forall\ x \in (0,1)$, we have

$$0 < T(x,x) < x,$$

the sequence $\left(x_T^{(n)}\right)_{n\epsilon N}$ is strictly decreasing.

Proposition 3.3.5 A t-norm T is strictly monotone if and only if the cancellation law holds, i.e. $T(x,y) = T(x,z)$ with $x > 0$ implies $y = z$.

Proof: Obviously, the strict monotonicity of T implies the validity of the cancellation law. Conversely, the strict monotonicity follows from the cancellation law together with the monotonicity (T-3).

Definition 3.3.6 A t-norm T is called Archimedian if $\forall\ (x,y) \in (0,1)^2\ \exists\ n \in N$ such that

$$x_T^{(n)} < y.$$

Notice that a left continuous Archimedian t-norm T is continuous.

Proposition 3.3.7 A t-norm T is Archimedian if and only if for each $x \in (0,1)$ we have

$$\lim_{n\to\infty} x_T^{(n)} = 0. \tag{3.3.7.1}$$

Proof: If T is Archimedian then $\forall\ x \in (0,1)$ and each $\epsilon > 0$, we have

$$0 \leq x_T^{(n_0)} < \epsilon \quad \text{for some } n_0 \in N$$

and, because of then monotonicity (T-3), we even have

$$0 \leq x_T^{(n)} < \epsilon \quad \forall n \geq n_0,$$

which inturn implies (3.3.7.1).
To prove the converse part, suppose that for a t-norm T and an arbitrary $x \in (0,1)$, we have

$$\lim_{n\to\infty} x_T^{(n)} = 0.$$

Then $\forall\ y \in (0,1), \exists\ n \in N$ such that

$$x_T^{(n)} < y,$$

which shows that T is Archimedian.

Theorem 3.3.8 (i) If T is an Archimedian t-norm then $\forall x \in (0,1)$, then

$$T(x,x) < x. \tag{3.3.8.1}$$

(ii) If T is a right continuous t-norm such that for all $x \in (0,1)$ the property (3.3.8.1) is satisfied, then T is Archimedian.

Definition 3.3.9 (i) A t-norm T is called nilpotent if it is continuous and (for each element $x \in (0,1)$) nilpotent, i.e., if $\exists$ some $n \in N$ such that $x_T^{(n)} = 0$.
(ii) An element $x \in (0,1)$ is called a zero divisor of T if there exists some $y \in (0,1)$ such that

$$T(x,y) = 0.$$

Obviously, each nilpotent element of a t-norm T is a zero divisor of T. For the t-norm T^{nM} each $x \in (0,1)$ is a zero divisor but each $x \in (0.5,1]$ is an idempotent element, and therefore it can not be a nilpotent element of T^{nM}. However, if a t-norm T has a zero divisor then it has also nilpotent elements. If

$$T(a,b) = 0 \quad \text{for some } a > 0 \text{ and } b > 0,$$

then for $c = \min\{a,b\} > 0$, we get

$$T(c,c) = 0.$$

A t-norm T has no zero divisors if and only if for all $x \in (0,1]$, $T(x,x) > 0$.

Theorem 3.3.10 Let T be a continuous Archimedian t-norm. Then the following are equivalent:
(i) T is nilpotent.
(ii) There exists some nilpotent element of T.
(iii) There exists some zero divisor of T.
(iv) T is not strict.

Proof: It is trivial to show that (i) implies (ii) whereas (ii) implies (iv). We already know that (ii) and (iii) are equivalent. In order to show that $(iv) \Rightarrow (i)$, assume that T is continuous and Archimedian but not strict. Then there are numbers $u, v, w \in [0,1]$ with $u > 0$ and $v < w$ such that $T(u,v) = T(u,w)$. Since we also have

$$T(v,w) \leq v < w = T(1,w)$$

the continuity of T implies that there is a $z \in [v,1)$ such that

$$v = T(z,w) = T(w,z).$$

Then we have

$$T(u,w) = T(u,v) = T(u,T(w,z)) = T(T(u,w),z)$$

and by induction, for each $n \in N$

$$T(u,w) = T\left(T(u,w), z_T^{(n)}\right)$$

Using the continuity and Proposition 3.3.7, we get

$$\begin{aligned} T(u,w) &= \lim_{n\to\infty} T\left(T(u,w), z_T^{(n)}\right) \\ &= T\left(T(u,w), \lim_{n\to\infty} z_T^{(n)}\right) \\ &= T(T(u,w), 0) = 0. \end{aligned}$$

Hence, u and v are zero divisors of T and then

$$b = \min\{u,w\} \in (0,1)$$

is a nilpotent element of T. Now for an arbitrary $a \in (0,1)$ the Archimedian property ensures that

$$a_T^{(n_0)} < b \qquad \text{for some } n_0 \in N$$

and, consequently,

$$0 \le a_T^{(2n_0)} \le b_T^{(2)} = 0,$$

showing that T is nilpotent.

3.4 Ordinal sums

A procedure of constructing a new t-norm from a system of given t-norms is based on an algebraic result from the semigroup theory.

Theorem 3.4.1([18, Theorem 1.16]) Let $(T_k)_{k\varepsilon K}$ be a family of t-norms and let $((\alpha_k,\beta_k))_{k\varepsilon K}$ be a family of pairwise disjoint open subintervals of the unit interval $[0,1]$ (i.e, K is an atmost countable index set). If the linear transformations $\{\Phi_k : [\alpha_k,\beta_k] \to [0,1]\}_{k\varepsilon K}$ is given by

$$\Phi_k(u) = \frac{u-\alpha_k}{\beta_k-\alpha_k}, \tag{3.4.1.1}$$

then the function $T : [0,1]^2 \to [0,1]$ defined by

$$T(x,y) = \begin{cases} \Phi_k^{-1}(T_k(\Phi_k(x), \Phi_k(y))) & \text{if } (x,y) \in (\alpha_k,\beta_k)^2, \\ \min(x,y) & \text{otherwise,} \end{cases} \tag{3.4.1.2}$$

is a triangular norm.

Proof: The function T is obviously commutative and fulfills the boundary conditions. Now we are left to show the monotonicity and associativity of T. To check the monotonicity of T, we have to show that the monotonicity of the function $T(x,.) : [0,1] \to [0,1]$ for all $x \in [0,1]$. If $x \notin \bigcup_{k \epsilon K} (\alpha_k, \beta_k)$ then $\forall\, y \in [0,1]$ we have

$$T(x,y) = \min\,(x,y),$$

therefore $T(x,.)$ is monotone. Otherwise, if $x \in (\alpha_{ko}, \beta_{ko})$ for some $ko \in K$ then we have

$$T(x,y) = \begin{cases} y & \text{if } y \in [0, \alpha_{ko}], \\ \Phi_k^{-1}(T_k(\Phi_k(x), \Phi_k(y))) & \text{if } y \in (\alpha_{ko}, \beta_{ko}), \\ x & \text{if } y \in [\beta_{ko}, 1]. \end{cases}$$

Now, for all $y \in (\alpha_{ko}, \beta_{ko})$ we have

$$\alpha_{ko} = \Phi_{ko}^{-1}(0) \leq T(x,y) \leq \Phi_k^{-1}(\min\ (\Phi_k(x), \Phi_k(y))) \leq x,$$

and the hence monotonicity of T_{ko} implies the monotonicity of $T(x,.)$, thus conclude the proof for monotonicity of T.
Note that the boundary condition of T_k imply

$$T(x,y) = \Phi_k^{-1}(T_k(\Phi_k(x), \Phi_k(y)))$$

not only for $(x,y) \in (\alpha_k, \beta_k)^2$, but also for $(x,y) \in [\alpha_k, \beta_k]^2$.
To show the associativity of T, we distinguish the following four possible cases.
Case I: If $x, y, z \in [\alpha_{ko}, \beta_{ko}]$ for some $ko \in K$, then the associativity of T_{ko} implies

$$T(x, T(y,z)) = T(T(x,y), z).$$

Case II: If $x, z \in [\alpha_{ko}, \beta_{ko}]$ and $y \notin [\alpha_{ko}, \beta_{ko}]$ for some $ko \in K$,then we have

$$\begin{aligned} T(x, T(y,z)) &= T(x, \min\,(y,z)) \\ &= \min\,(T(x,y), T(x,z)) \\ &= \min\,(\min\,(x,y), T(x,z)) \\ &= \min\,(y, T(x,z)) \\ &= \min\,(\min\,(y,z), T(x,z)) \\ &= \min\,(T\,(y,z), T(x,z)) \\ &= T(\min\,(x,y), z) \\ &= T(T(x,y), z), \end{aligned}$$

where the equalities follow from the definition and monotonicity of T.
Case III: If $x, y \in [\alpha_{ko}, \beta_{ko}]$ and $z \notin [\alpha_{ko}, \beta_{ko}]$ or ($y, z \in [\alpha_{ko}, \beta_{ko}]$ and $x \notin [\alpha_{ko}, \beta_{ko}]$) for some $ko \in K$. Now following the similar arguments as in Case II, one can show $T(x, T(y,z)) = T(T(x,y), z)$.

Case IV: In all the other cases we obtain

$$T(x, T(y, z)) = \min(x, y, z) = T(T(x, y), z).$$

This shows that T is associative and consequently, a t-norm.

Definition 3.4.2 Let $(T_k)_{k\in K}$ be a family of t-norms and let $((\alpha_k, \beta_k))_{k\in K}$ be a family of pairwise disjoint open subintervals of the unit interval $[0, 1]$ i.e, K is an atmost countable index set. Then the t-norm T defined by (3.4.2) is called the ordinal sum of summands $(< (\alpha_k, \beta_k), T_k >)_{k\in K}$, and we shall write

$$T = (< (\alpha_k, \beta_k), T_k >)_{k\in K}.$$

Example 3.4.3 (i) An empty ordinal sum of t-norm (i.e., an ordinal sum of t-norms with index set Φ,) yields the minimum T_M :

$$T_M = (\Phi) = (< (\alpha_k, \beta_k), T_k >)_{k\in\Phi}.$$

Example: (ii) Each t-norm T can be viewed as a trivial ordinal sum with one summand $< (0, 1), T >$:

$$T =< (0, 1), T > .$$

3.5 Representation of continuous t-norms

Definition 3.5.1 Let $g : [a, b] \longrightarrow [c, d]$ be strictly monotone bijection of $[a, b]$ onto $[c, d]$, where $[a, b]$ and $[c, d]$ are closed subintervals of the extended real line $[-\infty, +\infty]$. A monotone continuous mapping $g^{(-1)} : [-\infty, +\infty] \longrightarrow [a, b]$, such that $\forall\ y \in [c, d]$,
$g^{(-1)}(y) = g^{-1}(y)$,if $y > d$, and $g^{(-1)}(y) = g^{-1}(d)$ and if $y < c$, then $g^{(-1)}(y) = g^{-1}(c)$, is called a **pseudo-inverse** of g.

In respect of $g(x) = e^{x-1}$ on $[0, 1]$, its pseudo-inverse is given by

$$g^{(-1)}(x) = \begin{cases} 0, & if\ x < e^{-1} \\ \ln x + 1, & if\ x\varepsilon\ [e^{-1}, 1] \\ 1, & if\ x > 1. \end{cases}$$

We now extend the idea of the pseudo-inverse also for the general (not necessarily strictly) monotone function and which can be also not surjective.This extension will be very useful for the construction of new t-norms given in the coming Theorem(3.5.5).

Definition 3.5.2 Let $g : [a, b] \longrightarrow [c, d]$ be a monotne function from $[a, b]$ to $[c, d]$,where $[a, b]$ and $[c, d]$ are closed subintervals of the extended real line $[-\infty, +\infty]$. A mapping $g^{(-1)} : [c, d] \longrightarrow [a, b]$ defined by

$$g^{(-1)}(y) = \sup \{x\varepsilon\ [a, b] \mid (g(x) - y)(g(b) - g(a)) < 0\} \qquad (3.5.2.1)$$

is called a pseudo-inverse of g.

Remark 3.5.3 For a non-constant and non-decreasing function g equation (3.5.2.1) reduces to

$$g^{(-1)}(y) = sup\{x\varepsilon\ [a,b]\ \mid g(x) < y\}$$

Remark 3.5.4 For a non-constant and non-increasing function g equation (3.5.2.1) reduces to

$$g^{(-1)}(y) = \sup\ \{x\varepsilon\ [a,b] \mid g(x) > y\}$$

Using now the notion of pseudo-inverse (i.e. Definition 3.5.2), the following theorem is proved which furnishes a procedure to associate new t-norms to a given t-norm.

Theorem 3.5.5 Let $g : [0,1] \longrightarrow [0,1]$ be a non-decreasing function and T a t-norm such that

$$T(g(x), g(y)) \in \text{ Range}(g) \cup [0, g(0+)]\quad \forall x, y \in [0,1),$$

whereas $\forall x, y \in [0,1]$,

$$T(g(x), g(y)) \in \text{Range } (g),$$

$$\text{and } gog^{-1}(T(g(x), g(y))) = T(g(x), g(y)).$$

Then the function $T_g : [0,1]^2 \longrightarrow [0,1]$ defined by

$$T_g(x,y) = \begin{cases} g^{(-1)}(T(g(x), g(y))) & if\ x, y \in [0,1) \\ \min\ (x,y), & \text{othewise} \end{cases} \tag{3.5.5.1}$$

is a t-norm.

For a continuous function g and any t-norm T the above conditions are always satisfied and (3.5.5.1) always give a t-norm.

Definition 3.5.6 A function $T : [0,1]^2 \rightarrow [0,1]$ is a continuous Arhimedian triangular norm if and only if there exists a continuous, strictly decreasing function $t : [0,1] \longrightarrow [0,+\infty]$ with $t(1) = 0$ such that for all $x, y \in [0,1]$

$$T(x,y) = t^{-1}(\min(t(x) + t(y), t(0))) = t^{(-1)}(t(x) + t(y)). \tag{3.5.6.1}$$

The function t is called an **additive generator** of T; it is uniquely determined by T upto a positive multiplicative constant.

Proposition 3.5.7 If T be a continuous Archimedian t-norm, then

(i) T is nilpotent if and only if for each additive generator t of T we have $t(0) < +\infty$,
(ii) T is strict if and only if for each additive generator t of T we have $t(0) = +\infty$.

Proof: If t is an additive generator of T with $t(0) < +\infty$, then the element $x = t^{-1}\left(\frac{f(0)}{2}\right) \in (0,1)$ is nilpotent because of

$$x_T^{(2)} = t^{-1}(\min(2t(x), t(0))) = 0,$$

and therefore T is nilpotent t-norm.
If t is an additive generator of T with $t(0) = +\infty$ then for all $x, y \in (0,1]$ we have $t(x) + t(y) < t(0)$, i.e.,

$$T(x,y) = t^{-1}(t(x) + t(y)) > 0,$$

showing that T has no zero divisors. As a consequence of Theorem 3.3.10. The proof is complete.

Example 3.5.8 *(i)* A family of additive generators $\left(t_\lambda^F : [0,1] \to [0,+\infty]\right)_{\lambda \in (0,+\infty]}$ for the family $\left(T_\lambda^F\right)_{\lambda \in (0,+\infty]}$ of Frank t-norms is given by

$$t_\lambda^F(x) = \begin{cases} -\log x & \text{if } \lambda = 1, \\ 1 - x & \text{if } \lambda = +\infty, \\ -\log \frac{\lambda^x - 1}{\lambda - 1} & \text{otherwise.} \end{cases}$$

(ii) A family of additive generators $\left(t_\lambda^Y : [0,1] \to [0,+\infty]\right)_{\lambda \in (0,+\infty)}$ for the family $\left(T_\lambda^Y\right)_{\lambda \in (0,+\infty)}$ of Yager t-norms is given by

$$t_\lambda^Y(x) = (1-x)^\lambda.$$

(iii) For the family of Sugeno-Weber's t-co-norms $\left(S_\lambda^{SW}\right)_{\lambda \in (-1,\infty)}$, the corresponding additive generators are given by

$$S_\lambda^{SW}(x) = \begin{cases} x & \text{if } \lambda = 0, \\ \frac{\log(1+\lambda x)}{\log(1+\lambda)} & \text{if } \lambda \in (-1,\infty) \setminus \{0\}. \end{cases}$$

Definition 3.5.9 (Multiplicative generators) The representation of continuous Archimedian t-norms given in Definition 3.5.6 is based on the addition on the interval $[0,+\infty]$. There is a completely analogous representation based on the multiplication on $[0,1]$.

If $t : [0,1] \longrightarrow [0,\infty]$ is an additive generator of the continuous t-norm T, then we define strictly increasing function $\theta : [0,1] \longrightarrow [0,1]$ by $\theta(x) = e^{-t(x)}$, which is a multiplicative generator of T (duality between additive and multiplicative generators, see [27, 48]).

The following representation theorem holds.

Theorem 3.5.10 A function $T : [0,1]^2 \longrightarrow [0,1]$ is a continuous Archimedian triangular norm if and only if there exists a continuous, strictly increasing function $\theta : [0,] \longrightarrow [0,1]$ with $\theta(1) = 1$ (called multiplicative generator) such that for all $x, y \in [0,1]$

$$T(x,y) = \theta^{-1}(\max\ (\theta(x).\theta(y), \theta(0))) = \theta^{(-1)}(\theta(x)\theta(y)).$$

Moreover, T is strict t-norm if and only if every continuous multiplication generator θ of T is a nilpotent t-norm if and only if each continuous multiplicative generator θ of T satisfies $\theta(0) > 0$.

Proof: If $t : [0,1] \longrightarrow [0,+\infty]$ is an additive generator of a continuous Archimedian t-norm T, then $\theta : [0,1] \longrightarrow [0,1]$ given by

$$\theta(x) = e^{-t(x)}$$

obviously is a multiplicative generator of T.

The argument in the proof above can be reversed, if $\theta : [0,1] \longrightarrow [0,1]$ is a multiplicative generator of a continuous Archimedian t-norm T then $t : [0,1] \longrightarrow [0,+\infty]$ given by $t(x) = -\log\ (\theta(x))$ is an additive generator of T. As known, a multiplicative generator θ is uniquely determined by T upto a positive constant exponent.

Example 3.5.11 For strict T_P a multiplicative generator is given by $\theta(x) = x$. Whereas for nilpotent T_L a multiplicative generator is given by

$$\theta(x) = e^{x-1}.$$

If T is a t- norm with a multiplicative generator θ, then the function $\xi : [0,1] \longrightarrow [0,1]$ given by

$$\xi(x) = \theta(1-x)$$

is a multiplicative generator of the dual t-conorm S.

Now we shall need the following families of t-norms with the corresponding multiplicative generators, see [26].

Example 3.5.12 The Dombi family of t-norms $\left(T_\lambda^D\right)_{\lambda \in [0,\infty]}$ is defined by

$$T_\lambda^D(x,y) = \begin{cases} T_D(x,y) & \text{if } \lambda = 0 \\ T_M(x,y) & \text{if } \lambda = \infty, \\ \left(1 + \left(\left(\frac{1-x}{x}\right)^\lambda + \left(\frac{1-y}{y}\right)^\lambda\right)^{\frac{1}{\lambda}}\right)^{-1} & \text{if } \lambda \in (0,\infty). \end{cases}$$

T_λ^D is Archimedian if and only if $\lambda \in [0, \infty)$. A family of multiplicative generators $\left(\theta_\lambda^D\right)_{\lambda \in (0,\infty)}$ is given by

$$\theta_\lambda^D(x) = e^{-\left(\frac{1-x}{x}\right)^\lambda}.$$

T_λ^D is strict if and only if $\lambda \in (0, \infty)$.

Example 3.5.13 The Schweizer-Sklar family of t-norms $\left(T_\lambda^{SS}\right)_{\lambda \epsilon[-\infty,+\infty]}$ is defined by

$$T_\lambda^{SS}(x,y) = \begin{cases} T_M(x,y) & \text{if } \lambda = -\infty, \\ \left(x^\lambda + y\lambda - 1\right)^{\frac{1}{\lambda}} & \text{if } \lambda \epsilon\, (-\infty, 0), \\ T_P(x,y) & \text{if } \lambda = 0, \\ \left(\max\left(x^\lambda + y^\lambda - 1\right), 0\right)^{\frac{1}{\lambda}} & \text{if } \lambda \in (0,\infty), \\ T_D(x,y) & \text{if } \lambda = \infty. \end{cases}$$

T_λ^{SS} is continuous Archimedian if and only if $\lambda \in (-\infty, +\infty)$. A family of multiplicative generators $\left(\theta_\lambda^{SS}\right)_{\lambda \in (-\infty,+\infty)}$ is given by

$$\theta_\lambda^{SS}(x) = \begin{cases} x & \text{if } \lambda = 0, \\ e^{\frac{\left(x^\lambda - 1\right)}{\lambda}} & \text{if } \lambda \in (-\infty,0) \cup (0,\infty). \end{cases}$$

T_λ^{SS} is strict if and only if $\lambda \in (-\infty, 0]$.

Example 3.5.14 The Aczel-Alsina family of y-norms $\left(T_\lambda^{AA}\right)_{\lambda \in [0,\infty]}$ is defined by

$$T_\lambda^{AA}(x,y) = \begin{cases} T_D(x,y) & \textit{if } \lambda = 0 \\ T_M(x,y) & \textit{if } \lambda = \infty \\ e^{-\left(|\log x|^\lambda + |\log y|^\lambda\right)^{\frac{1}{\lambda}}} & \textit{if } \ \lambda \in (0,\infty). \end{cases}$$

T_λ^{AA} is Archimedian if and only if $\lambda \in [0, \infty)$. A family of multiplicative generators $\left(\theta_\lambda^{AA}\right)_{\lambda \in (0,\infty)}$ is given by

$$\theta_\lambda^{AA}(x) = e^{-(-\log x)^\lambda}.$$

T_λ^{AA} is strict if and only if $\lambda \in (0, \infty)$.

Proposition 3.5.15 Let T be a continuous Archimedian t-norm and $t : [0,1] \longrightarrow [0, +\infty]$ an additive generator of T. If T^* is a t-norm which is isomorphic to T, i.e., if there is a strictly increasing bijection $\Phi : [0,1] \longrightarrow [0,1]$ such that for all $x, y \in [0,1]$

$$T^*(x,y) = \Phi^{-1}\left(T\left(\Phi(x), \Phi(y)\right)\right),$$

then T^* is also a continuous Archimedian t-conorm, and the function $h : [0,1] \longrightarrow [0,+\infty]$ defined by

$$h = t \circ \Phi$$

is an additive generator of T^*.

Proof: Taking into account $h(0) = t(\Phi(0)) = t(0)$, we obtain

$$T^*(x,y) = \Phi^{-1} \circ t^{-1}(\min(t \circ \Phi(x) + t \circ \Phi(y), t(0)))$$

$$= h^{-1}(\min(h(x) + h(y), h(0))) \qquad \forall x, y \in [0,1].$$

It is straightforward that each isomorphism $\Phi : [0,1] \longrightarrow [0,1]$ preservers the other algebraic and analytic properties of t-norms. In particular, each t-norm which is isomorphic to a strict or nilpotent t-norm, is strict or nilpotent, respectively.

Proposition 3.5.16 (i) Any two strict t-norms are isomorphic.
(ii) Any two nilpotent t-norms are isomorphic.

Proof: If T_1 and T_2 are two strict or two nilpotent t-norms with additive generators t_1 and t_2, respectively, such that $t_1(0) = t_2(0)$ Observe that this condition is always satisfied in the case of additive generators of strict t-norms, then in both cases the function Φ : $[0,1] \longrightarrow [0,1]$ defined by $\Phi = t_1^{-1} \circ t_2$ is an isomorphism between T_2 and T_1.As known, a multiplicative generator θ is uniquely determined by T upto a positive constant exponent.

3.6 Triangular norms with left-continuous diagonals at (1,1)

We know that a t-norm T is left continuous if for every sequence $(x_n)_{n\epsilon N} \subseteq [0,1]$ and every $y \in [0,1]$ we have

$$\sup_{n\epsilon N} T(x_n, y) = T\left(\sup_{n\epsilon N} x_n, y\right).$$

In our subsequent discussion, special attention will be taken on the continuity in the point $(1,1)$. It is obvious that the left continuity in the point $(1,1)$ implies

$$\sup_{x\epsilon[0,1)} T(x,x) = 1. \qquad (3.6.1)$$

But its converse is not true. The property (3.6.1) means that the diagonal section δ_T : $[0,1] \longrightarrow [0,1]$ of a t-norm T, given by

$$\delta_T(x) = T(x,x),$$

is left continuous at the point 1. We remark that generally even the continuity of the whole diagonal $\delta_T(x) = T(x,x)$, $x \in [0,1]$, of an Archimedian t-norm T does not imply the continuity of the t-norm T. The t-norm T defined by

$$T^{\Delta}(x,y) = \begin{cases} \frac{xy}{2} & \text{if } x,y \in [0,1), \\ \min(x,y) & \text{othewise.} \end{cases}$$

is strictly monotone continuous for $(x,y) \in [0,1)$ but it does not satisfy the condition (3.6.1). Hence it is not continuous in the point $(1,1)$.
The diagonal section $\delta_T : [0,1] \to [0,1]$ of a t-norm T characterizes some algebraic properties of a t-norm T, e.g., (i) the Archimedian property, (ii) idempotent elements are fixed points of δ_T, and (iii) The only continuous t-norm which is uniquely determined by its function δ_T is the minimum T_M.

We now state a general notion of an additive generator even for non-continuous t-norms.

Definition 3.6.1 An additive generator $t : [0,1] \to [0,\infty]$ (if it exists) of a t-norm T is a strictly decreasing function such that:
(i) t is right continuous at 0 with $t(1) = 0$;
(ii) for $x,y \in [0,1]$ we have $t(x) + t(y) \varepsilon$ Range $(t) \cup [t(0+), \infty]$;
(iii) $T(x,y) = t^{(-1)}(t(x) + t(y))$, $x,y \in [0,1]$.

For example, the non-continuous t-norm T_D has as additive generator

$$t(x) = \begin{cases} 2-x & \text{if } x\varepsilon\,[0,1) \\ 0 & \text{if } x = 1, \end{cases}$$

whereas the non-continuous t-norm T^{Δ} $(given)$ has an additive generator

$$t(x) = \begin{cases} -\log\frac{x}{2} & \text{if } x\varepsilon\,[0,1), \\ 0 & \text{if } x = 1. \end{cases}$$

But, for a t-norm T having an additive generator is a strong property, as continuity only at the point $(1,1)$ (together with (3.61)) implies the complete continuity of T.

Theorem 3.6.2 Every left-continuous t-norm in the point $(1,1)$ with an additive generator always continuous.

Proof: We shall show that for a left-continuous t-norm in the point $(1,1)$ T (with an additive generator t) the additive generator is continuous. First we shall prove that t is left continuous in 1. Suppose the contrary, i.e., $t(1-0) > 0$. Therefore

$$T(1-0, 1-0) = t^{(-1)}(2t(1-0)) < 1,$$

a contradiction with the left-continuity of T in the point $(1,1)$. Now we shall prove that t is continuous on the interval $(0,1]$. Suppose the contrary. Then there is some interval $(c,d) \subset (0,\, t(0))$ such that

$$(c,d) \cap \text{Range } (t) = \Phi.$$

Therefore $\forall\ n \in N$ we have also

$$\left(\frac{c}{n}, \frac{d}{n}\right) \cap Range\,(t) = \Phi.$$

If n_0 is the smallest integer greater than $\frac{c}{(d-c)}$ then we have $\frac{c}{n} < \frac{d}{(n+1)}$ for all $n \geq n_0$,i.e., the intervals $\left(\frac{c}{n}, \frac{d}{n}\right)$ and $\left(\frac{c}{n+1}, \frac{d}{n+1}\right)$ are overlapping. Hence

$$\bigcup_{n=n_0}^{\infty} \left(\frac{c}{n}, \frac{d}{n}\right) = \left(0, \frac{d}{n_0}\right).$$

Therefore $t\,(1-0) \geq \frac{d}{n_0}$,
a contradiction with the left continuity of t in 1.

Theorem 3.6.3 A function $T : [0,1]^2 \rightarrow [0,1]$ is a continuous t-norm if and only if T is an ordinal sum of continuous Archimedian t-norms.

Proof: For the proof, one is referred to Hadžić and Pap [18].

3.7 Triangular norms of H-type

Definition 3.7.1 A t-norm T is of H-type if the family $\left(x_T^{(n)}\right)_{n \in N}$ is equi-continuous at the point $x = 1$, where $x_T^{(n)}$ is defined by

$$x_T^{(1)} = x,\ x_T^{(n)} = T\left(x_T^{(n-1)}, x\right), \quad n \geq 2, x \in [0,1].$$

There do exist a non trivial example of a t-norm T such that $\left(x_T^{(n)}\right)_{n \in N}$ is an equi-continuous family at the point $x = 1$.

Example 3.7.2 Let T be a continuous t-norm and let for every $m \in N \cup \{0\}$:

$$I_m = \left[1 - 2^{-m}, 1 - 2^{-m-1}\right].$$

If

$$T\,(x,y) = 1 - 2^{-m} + 2^{-m-1}T\left(2^{m+1}\left(x - 1 + 2^{-m}\right), 2^{m+1}\left(\left(y - 1 + 2^{-m}\right)\right)\right)$$

for $(x,y) \in I_m \times I_m$ and

$$T(x,y) = \min(x,y) \ for\ (x,y) \notin \cup_{m \in N \cup \{0\}} I_m \times I_m,$$

then the family $\left(x_T^{(n)}\right)_{n \in N}$ is an equi-continuous family at the point $x = 1$, i.e., T is a t -norm of H-type.

Proposition 3.7.3 An Archimedian continuous t-norm T cannot be of H-type.

Proof: The reason is that for every $x, y \in (0,1)$ there exists $n \in N$ such that $x_T^{(n)} < y$, and therefore the family $\left(x_T^{(n)}\right)_{n \in N}$ can not be equi-continuous at the point $x = 1$.Hence if in the ordinal sum representation

$$T = (<(\alpha_k, \beta_k), T_k>)_{k \in K} \tag{3.7.3.1}$$

there is some $k^* \in K$ such that $\beta_{k^*} = 1$, then T_{k^*} is Archimedian on $(\alpha_{k^*}, 1)$ and

$$T = (<(\alpha_k, \beta_k), T_k>)_{k \in K}$$

cannot be H-type.

Proposition 3.7.4 A continuous t-norm T is of H-type if and only if

$$T = (<(\alpha_k, \beta_k), T_k>)_{k \in K} \ \ and\ sup\,\beta_k < 1 \ \ or\ \ sup\,\alpha_k = 1.$$

Proof: In view of Theorem 3.6.3, T is continuous if and only if

$$T = (<(\alpha_k, \beta_k), T_k>)_{k \in K}$$

with Archimedian continuous summands T_k, $(k \in K)$.

Suppose that T is of H-type, then $\beta_k < 1$ for every $k \in N$. Since the system $((\alpha_k, \beta_k))_{k \in K}$ is disjoint we have that

$$\sup \beta_k < 1 \ \text{ or } \ \sup \beta_k = \sup \alpha_k = 1.$$

The case $\sup \beta_k = 1$ with $\sup \alpha_k < 1$ is impossible, as

$$sup\,\beta_k = \max\ \beta_k = \beta_{k^*} = 1,$$

for some $k^* \in K$, which leads to a contradiction.

Now we are left to show that T is of H-type, if

$$sup\,\beta_k < 1 \; or \; sup\,\alpha_k = 1$$

In case $\sup \beta_k < 1$ then $x_T^{(n)} = x$, for any $x > sup\,\beta_k$, and therefore T is of H-type. Otherwise let $\sup \alpha_k = 1$,then for any $\epsilon > 0$ there is a $k \in K$ such that

$$1 - \epsilon < \alpha_k$$

and so (for $x > \alpha_k$,)

$$x_T^{(n)} \geq \alpha_k > 1 - \epsilon$$

which shows that T is of H-type.

Theorem 3.7.5 In respect of a t-norm T, the following hold:
(i) Suppose that there exists a strictly increasing sequence $(b_n)_{n \in N}$ from the interval $[0, 1)$ such that

$$\lim_{n \to \infty} b_n = 1 \text{ and } T(b_n, b_n) = b_n.$$

Then T is of H-type.

(ii) If T is continuous and H-type, then there exists a sequence $(b_n)_{n \in N}$ meeting the requirements of (i).

Proof:(i) Let $\lambda \in (0, 1)$ be given. We shall prove that there exists $\eta \in (0, 1)$ such that $x > 1 - \eta$ implies that

$$x_T^{(n)} > 1 - \lambda \quad \forall n \in N.$$

Since $\lim_{n \to \infty} b_n = 1$, there exists $k = k(\lambda) \in N$ such that $b_k > 1 - \lambda$. Let $\eta = 1 - b_k$. if $x > 1 - \eta = b_k$ then

$$T(x, x) \geq T(b_k, b_k) = b_k > 1 - \lambda$$

which amounts to say that T is of H-type.

(ii) Suppose now that T is of $H-$type. Let k be a fixed natural number, $\epsilon_k = \frac{1}{k}$ and let η_k be such that $(\forall n \in N)$

$$x > 1 - \eta_k \Rightarrow x_T^{(n)} > 1 - \epsilon_k\,.$$

The sequence $\left(x_T^{(n)}\right)_{n \in N}$ $(x > 1 - \eta_k)$ is non-decreasing and therefore there exists $b_k = \lim_{n \to \infty} x_T^{(n)}$, $k \in N$.
Since

$$x_T^{(2n+1)} = T\left(x_T^{(n)}, x_T^{(n)}\right),$$

using the continuity of T we obtain that

$$\lim_{n\to\infty} x_T^{(2n+1)} = b_k = T\left(\lim_{n\to\infty} x_T^{(n)}, \lim_{n\to\infty} x_T^{(n)}\right) = T(b_k, b_k) \quad \forall k \in N.$$

which completes the proof.

From the proof of the above theorem we see the condition of continuity of the whole sequence $\left(x_T^{(n)}\right)_{n\in N}$ can be replaced by the condition that the function

$$\delta_T(x) = T(x,x) \ \ (x \in [0,1])$$

is right continuous on the interval $[b,1)$ for $b < 1$.

Theorem 3.7.6 For a continuous t-norm T, the following are equivalent:
(a) T is not of H-type.
(b) There exists $a_T \in [0,1)$ and a continuous strictly increasing and surjective mapping $\Phi_{a_T} : [a_T, 1] \to [0,1]$ such that

$$T(x,y) = \Phi_{a_T}^{-1}\left(\Phi_{a_T}(x) * \Phi_{a_T}(y)\right), \quad \forall x, y \geq a_T,$$

where the operation $*$ is either T_P or T_L.

Proof: A proof of this theorem is available in Hadžić and Pap [18]. Due to the limitation of the space, we have omitted the proof.

Definition 3.7.7 Let T be a t-norm. We say that $T \in H$ if and only if there exists a non decreasing sequence $(b_n)_{n\in N}$ from $(0,1)$ such that $\lim\limits_{n\to\infty} b_n = 1$ ($\forall n \in N$) such that

$$1 \geq x > b_n, 1 \geq y > b_n \Rightarrow T(x,y) > b_n.$$

One may note that if a t-norm T is of H-type and summands in the representing ordinal sum form are strict then $T \in H$

3.8 Domination of t-norms

The domination relation in the family of t-norms arises in the construction of the cartesian product of probabilistic metric spaces. We have by [47], see also [26, 61].

Definition 3.8.1 A t-norm T_2 dominates a t-norm $T_1, T_2 \gg T_1$, if for $a, b, c, d \in [0,1]$,

$$T_2(T_1(a,b), T_2(c,d)) \geq T_1(T_2(a,c), T_2(b,d)). \tag{3.8.1.1}$$

Example 3.8.2 It is straightforward to note that for any t-norm T , $T_M \gg T$ and $T \gg T_D$.

Subsequent, the following characterization of the domination relation will be very useful in our investigations.

Theorem 3.8.3 A strict t-norm T_2 (with a generator t_2) dominates a strict t-norm T_1 (with a generator t_1) if and only if the function $h = t_2 \circ t_1^{-1}$ satisfies the following inequality

$$h^{-1}\left(h\left(x,y\right)+h\left(z+w\right)\right) \leq h^{-1}\left(h\left(x\right)+h\left(z\right)+h^{-1}\left(h\left(y\right)+h\left(w\right)\right)\right) \qquad (3.8.3.1)$$

for arbitrary non-negative real numbers x, y, z and w.

Proof: Since $t_1^{-1} : [0, +\infty] \to [0, 1]$ it follows for arbitrary non-negative real numbers x, y, z and w that

$$a = t_1^{-1}(x), b = t_1^{-1}(z), c = t_1^{-1}(y), \text{and } d = t_1^{-1}(w)$$

belong to the interval $(0, 1]$.
For these values the inequality (3.8.1.1) yields the form

$$T_2\left(T_1\left(t_1^{-1}(x), t_1^{-1}(y)\right), T_1\left(t_1^{-1}(z), t_1^{-1}(w)\right)\right) \geq T_2\left(T_1\left(t_1^{-1}(x), t_1^{-1}(z)\right), T_1\left(t_1^{-1}(y), t_1^{-1}(w)\right)\right)$$

Using the representations of strict t-norms T_1 and T_2 by their generators $t_1$and t_2 respectively, we obtain

$$\begin{aligned} & t_2^{-1}\left(\left(t_2 \circ t_1^{-1}\right)(x+y) + \left(t_2 \circ t_1^{-1}\right)(z+w)\right) \\ \geq\ & t_1^{-1}\left(\left(t_1 \circ t_2^{-1}\right)\left(t_2 \circ t_1^{-1}\right)(x) + \left(t_2 \circ t_1^{-1}\right)(z) + \left(t_2 \circ t_1^{-1}\right)\left(t_2 \circ t_1^{-1}\right)(y) + \left(t_2 \circ t_1^{-1}\right)(w)\right) \end{aligned}$$

Since by the representation of T_1 the function t_1 is strictly decreasing applying it on both sides of the preceding inequality we obtain the inequality (3.8.3.1).
Now, let a, b, c and d be arbitrary real numbers from the interval $[0, 1]$. Then since $t_1 : [0, 1] \to [0, \infty]$, we obtain that $x = t_1(a)$, $y = t_1(c)$, $z = t_1(b)$, and $w = t_1(d)$ are non-negative real numbers. Putting these values in (3.8.3.1) we obtain

$$\begin{aligned} & \left(t_1 \circ t_2^{-1}\right)\left(\left(t_2 \circ t_1^{-1}\right)(t_1(a) + t_1(c)) + \left(t_2 \circ t_1^{-1}\right)\right)(t_1(b) + t_1(d)) \\ \leq\ & \left(t_1 \circ t_2^{-1}\right)\left(t_2 \circ t_1^{-1}\right)\left(t_1(a) + \left(t_2 \circ t_1^{-1}\right)(t_1(b))\right) + \left(t_1 \circ t_2^{-1}\right)\left(\left(t_1(c) + \left(t_2 \circ t_1^{-1}\right)t_1(d)\right)\right). \end{aligned}$$

If we rewrite the last inequality using representation theorem we obtain (3.8.1.1) i.e. that $T_2 \gg T_1$.

Proposition 3.8.4 If a t-norm T_2 dominates a t-norm T_1, then T_1 is weaker than T_2. The opposite is not true even for strict t-norms.

Proof: For the proof, one may consult Hadžić and Pap et.al [18].

Proposition 3.8.5 The relatiion $\gg$ between t-norms is reflexive and anti-symmetric.

Proof: Using the associativity and commutativity of a t-norm T we have

$$T\left(T\left(a,b\right),T\left(c,d\right)\right)=T\left(T\left(a,c\right),T\left(b,d\right)\right),$$

i.e., $\gg$ is reflexive.
The relation $\gg$ is anti-symmetric: Suppose that $T_1 \gg T_2$ and $T_2 \gg T_1$. Then by Proposition 3.8.4 we have $T_1 \geq T_2$ and $T_2 \geq T_1$, which implies $T_1 = T_2$.

Proposition 3.8.6 If a strictly increasing continuous function $h:[0,\infty)\rightarrow[0,\infty]$, with $h\left(0\right)=0$,satisfies 3.8.3.1 then it is convex and super-additive. Its converse is not true, i.e., continuous increasing convex functions h with $h\left(0\right)=0$ need not satisfy $(3.8.3.1)$.

Proof: For arbitrary but fixed no-negative real numbers a and b, such that $a \leq b$, we take

$$x=h^{-1}\left(a\right),y=h^{-1}\left(\frac{a+b}{2}\right)-h^{-1}\left(a\right),z=h^{-1}\left(\frac{b-a}{2}\right),\text{ and } w=0.$$

Putting these values in (3.8.3.1) we obtain

$$\frac{h^{-1}\left(a\right)+h^{-1}\left(b\right)}{2}\leq h^{-1}\left(\frac{a+b}{2}\right).$$

Hence h^{-1} is a concave function and therefore $h$is convex function. Hence by using $h\left(0\right)=0$, h is subadditive. To disprove the converse, one may use the following example

$$h\left(x\right)=\begin{cases} x, & \text{if } 0\leq x\leq 1,\\ 2x-1, & \text{if } x\geq 1.\end{cases}$$

Notice that for $x=z=w=1$ and $y=\frac{1}{2}$, 3.8.3.1 is not satisfied.

3.9 Countable extension of t-norms

Each t-norm T can be extended (by associativity) in a unique way to an n-any operation taking for$(x_1,x_{2,}x_3,....x_n)\in\ [0,1]^n$,$n\ \in N$, the values $T(x_1, x_{2,}x_3, ...x_3)$ which is defined by

$$\mathsf{T}_{i=1}^{0} x_i = 1,\ \mathsf{T}_{i=1}^{n} x_i = T\left(\mathsf{T}_{i=1}^{n-1} x_i, x_n\right) = T(x_1, x_{2,}x_3, ...x_n).$$

Specially, we have

$$\begin{aligned} T_L(x_1, x_{2,}x_3, ...x_n) &= \max\left(\sum_{i=1}^{n} x_i - (n-1), 0\right) \\ \text{and } T_M(x_1, x_{2,}x_3, ...x_n) &= \min(x_1, x_{2,}x_3, ...x_n). \end{aligned}$$

We can extend T to a countable infinitary operation taking for any sequence $(x_n)_{n\epsilon N}$ from $[0,1]$ the values

$$\mathsf{T}_{i=1}^{\infty} x_i = \lim_{n\to\infty} \mathsf{T}_{i=1}^{n} x_i. \tag{3.9.1}$$

The limit on the right side of (3.9.1) exists since the sequence $(\mathsf{T}_{i=1}^{n} x_i)_{n\in N}$ is non-increasing and bounded from below.

In the fixed point theory it is of interest to investigate the classes of t-norms T and sequences $(x_n)_{n\epsilon N}$ from $[0,1]$ such that

$$\lim_{n\to\infty} \mathsf{T}_{i=n}^{\infty} x_i. = \lim_{n\to\infty} \mathsf{T}_{i=1}^{\infty} x_{n+i} = 1. \tag{3.9.2}$$

In the classical case $T = (T_P)$ we have

$$(T_P)_{i=1}^{n} = \prod_{i=1}^{n} x_i$$

and for every sequence $(x_n)_{n\epsilon N}$ from $[0,1]$ with $\sum_{i=1}^{n}(1-x_i) < \infty$, it follows that

$$\lim_{n\to\infty} (T_P)_{i=n}^{\infty} = \lim_{n\to\infty} \Pi_{i=1}^{n} x_i = 1.$$

Namely, it will known that

$$\prod_{i=1}^{\infty} x_i > 0 \Leftrightarrow \lim_{n\to\infty} \Pi_{i=n}^{\infty} x_i = 1 \Leftrightarrow \sum_{i=1}^{n} (1-x_i) < \infty.$$

The equivalence

$$\sum_{i=1}^{\infty} (1-x_i) < \infty \Leftrightarrow \lim_{n\to\infty} \mathsf{T}_{i=n}^{\infty} x_i. = 1. \tag{3.9.3}$$

holds also for $T \geq T_L$.

Proposition 3.9.1 If $\{x_n\} \subset [0,1]$ such that $\lim_{n\to\infty} x_n = 1$ and the t-norm T is of H-type, then (3.9.2) holds.

Proof: Since t-norm T is of H-type for every $\lambda \in (0,1)$ there exists $\delta(\lambda) \in (0,1)$ such that

$$x \geq \delta(\lambda) \Rightarrow \mathrm{T}_{i=1}^{p} x > 1 - \lambda$$

for every $p \in N$. Since $\lim_{n\to\infty} x_n = 1$ there exists $n_0(\lambda) \in N$ such that $x_n \geq \delta(\lambda)$ for every $n \geq n_0(\lambda)$. Hence

$$\begin{aligned} \mathrm{T}\ {}_{i=1}^{p} x_{n+i} &\geq \mathrm{T}_{i=1}^{p} \delta(\lambda) \\ &> 1 - \lambda, \end{aligned}$$

for every $n \geq n_0(\lambda)$ and every $p \in N$. This means that (3.9.2) holds.

Lemma 3.9.2 Let T be a strict t-norm with an additive generator t, and the corresponding multiplicative generator θ.Then we have

$$\mathrm{T}\ {}_{i=1}^{\infty} x_i = t^{-1}\left(\sum_{i=1}^{\infty} t(x_i)\right)$$
$$\mathrm{T}\ {}_{i=1}^{\infty} x_i = \theta^{-1}\left(\Pi_{i=1}^{\infty} \theta(x_i)\right)$$

Proposition 3.9.3 Let T be a strict t-norm with an additive generator t, and the corresponding multiplicative generator θ. For a sequence $(x_n)_{n\epsilon N}$ form the interval $(0,1)$ such that $\lim_{n\to\infty} x_n = 1$
the condition

$$\lim_{n\to\infty} \sum_{i=n}^{\infty} t(x_i) = 0,$$

or the condition

$$\lim_{n\to\infty} \prod_{i=n}^{\infty} \theta(x_i) = 1,$$

holds if and only if (3.9.2) is satisfied.

Example 3.9.4 Let $\left(T_\lambda^D\right)_{\lambda\epsilon(0,\infty)}$ be a Dombi family of t-norms and $(x_n)_{n\epsilon N}$ be a sequence of elements from $(0,1]$ such that $\lim_{n\to\infty} x_n = 1$. Then

$$\sum_{i=1}^{\infty}\left(\frac{1-x_i}{x_i}\right)^{\lambda} < \infty \Leftrightarrow \lim_{n\to\infty}\left(T_\lambda^D\right)_{i=n}^{\infty} x_i = 1.$$

For a t-norm $T_\lambda^D, \lambda \in (0,\infty)$, the multiplicative generator θ_λ^D is given by

$$\theta_\lambda^D = e^{-1\left(\frac{1-x_i}{x_i}\right)^{\lambda}}$$

and therefore with the property $\theta_\lambda^D(1) = 1$. Hence

$$\sum_{n=1}^{\infty}(1-x_n)^{\lambda} < \infty \Leftrightarrow \sum_{n=1}^{\infty}\left(\frac{1-x_n}{x_n}\right)^{\lambda} < \infty.$$

and

$$\sum_{n=1}^{\infty}(1-x_n)^{\lambda}<\infty \Leftrightarrow \lim_{n\to\infty}\left(T_{\lambda}^{D}\right)_{i=n}^{\infty}x_i=1.$$

Proposition 3.9.5 If $\{x_n\}\subset(0,1)$ such that the series $\sum\limits_{n=1}^{\infty}(1-x_n)$is convergent. Then $\forall\,\lambda\in(-1,\infty]$

$$\lim_{n\to\infty}\left(T_{\lambda}^{SW}\right)_{i=n}^{\infty}x_i=1.$$

Proof: An additive generator of T_{λ}^{SW} for $\lambda\in(-1,0)$ is given by

$$t_{\lambda}^{SW}(x)=-\log\left(\frac{1+\lambda x}{1+\lambda}\right).\frac{1}{\log(1+\lambda)}.$$

We shall prove that for some $n_1\in N$ and every $p\in N$.

$$\prod_{i=1}^{p}\theta_{\lambda}^{SW}(x_{n+i-1})=\exp\left(\sum_{i=1}^{p}\log\left(\frac{1+\lambda x_{n+i-1}}{1+\lambda}\right).\frac{1}{\log(1+\lambda)}\right)>e^{-1}. \tag{3.9.5.1}$$

for every $n\geq n_1$ since in this case

$$\left(T_{\lambda}^{SW}\right)_{i=1}^{p}x_{n+i-1}=\left(\theta_{\lambda}^{SW}\right)^{-1}\left(\prod_{i=1}^{p}\theta_{\lambda}^{SW}(x_{n+i-1})\right). \tag{3.9.5.2}$$

We have to prove that for some $n_1\in N$ and every $p\in N$.

$$-\frac{1}{\log(1+\lambda)}\sum_{i=0}^{p}\log\left(\frac{1+\lambda x_{n+i-1}}{1+\lambda}\right)<1\quad\forall n\geq n_1, \tag{3.9.5.3}$$

since (3.9.5.3) implies (3.9.5.1). From

$$\lim_{n\to\infty}(1-x_n)=0,$$

it follows that

$$\log\left(1+\frac{\lambda}{1+\lambda}(x_n-1)\right)\sim\frac{\lambda}{1+\lambda}(x_n-1)$$

and therefore the series

$$-\frac{1}{\log(1+\lambda)}\sum_{n=1}^{\infty}\log\left(1+\frac{\lambda}{1+\lambda}(x_n-1)\right)$$

is convergent. Hence it follows that there exists $n_1\in N$ such that (3.9.5.1) holds for every $n\geq n_1$ and every $p\in N$, and this implies (3.9.5.2).

The above proposition holds also for $\lambda\geq 0$ since in this case $T_{\lambda}^{SW}\geq T_L$.

It is of special interest for the fixed point theory in probabilistic metric spaces to investigate condition (3.9.2) for a special sequence $(1-q^n)_{n\in N}$ for $q \in (0,1)$.

Definition 3.9.6 We say that for some $q \in (0,1)$ a t-norm T is q-convergent if $\lim\limits_{n\to\infty} \mathrm{T}_{i=n}^{\infty}(1-q^i) = 1$.
Since

$$\lim_{n\to\infty}(1-q^n) = 1 \text{ and } \sum_{n=1}^{\infty}(1-(1-q^n))^s < \infty \qquad \forall s > 0$$

it follows that all t-norms from the family

$$\mathcal{T}_o = \cup_{\lambda\in(0,\infty)}\left\{T_\lambda^D\right\} \cup \bigcup_{\lambda\in(0,\infty)}\left\{T_\lambda^{AA}\right\} \cup \mathcal{T}^H \bigcup_{\lambda\in(-1,\infty]}\left\{T_\lambda^{SW}\right\}$$

are q-convergent for every $q \in (0,1)$.

The following example shows that not every strict t-norm is q-convergent.

Example 3.9.7 Let T be the strict t-norm with an additive generator $t(x) = -\frac{1}{\log(1-x)}$.
In this case the series $\sum\limits_{i=1}^{\infty} t(1-q^i)$ for any $q \in (0,1)$ is not convergent as

$$\sum_{i=1}^{\infty} t\left(1-q^i\right) = -\sum_{i=1}^{\infty}\frac{1}{\log\left(q^i\right)} = -\sum_{i=1}^{\infty}\frac{1}{i\log\, q}.$$

In the following two propositions we shall give sufficient conditions for a t-norm T to be q-convergent.

Proposition 3.9.8 Let T and T_1 be strict t-norms and t and t_1 their additive generators, respectively, and there exists $b \in (0,1)$ such that $t(x) \le t_1(x) \quad \forall x \varepsilon\ (b,1]$. If T_1 is q-convergent, then T is q-convergent.

Proof: Since T_1 is q-convergent we have

$$\lim_{n\to\infty}(T_1)_{i=n}^{\infty}\left(1-q^i\right) = 1,$$

which in turn yields

$$\lim_{n\to\infty}\sum_{i=n}^{\infty} t_1\left(1-q^i\right) = 0. \tag{3.9.8.1}$$

Since there exists $n_0 \in N$ such that $1-q^{no} \in (b,1]$ we have

$$t(1-q^n) \le t_1(1-q^n) \quad \forall n \ge n_0.$$

Therefore, by (3.9.8.1)

$$\lim_{n\to\infty}\sum_{i=n}^{\infty} t_1\left(1-q^i\right)=0,$$

i.e., T is q-convergent.

Proposition 3.9.9 If T is a strict t-norm with a generator t having a bounded derivative on the interval $(b,1)$ for some $b\in(0,1)$, then T is q-convergent for every $q\in(0,1)$.

Proof: By the Lagrange mean value theorem we have for every $x\in(b,1)$ that

$$t(x)-t(1)=t(x)=t'(\xi)(x-1)$$

for some $\xi\in(x,1)$, and therefore

$$\sum_{i=i_o}^{\infty} t\left(1-q^i\right)\le M\sum_{i=i_o}^{\infty} q^i,$$

where

$$M=\sup_{x\epsilon(b,1)}\left|t'(x)\right|,\quad \text{and } 1-q^{io}\in(b,1).$$

Proposition 3.9.10 Let T be a t-norm and $\Phi:(0,1]\to[0,\infty)$. If for some $\delta\in(0,1)$ and every $x\in[0,1]$, $y\in[1-\delta,1]$

$$|T(x,y)-T(x,1)|\le\Psi(y) \tag{3.9.10.1}$$

then for every sequence $\{x_n\}\subset[0,1]$ such that $\lim_{n\to\infty}x_n=1$ the following implication holds:

$$\sum_{n=1}^{\infty}\Psi(x_n)<\infty\Rightarrow\lim_{n\to\infty}\left(\mathrm{T}_{i=n}^{\infty}x_i-x_n\right)=0. \tag{3.9.10.2}$$

Proof: Let $n_0(\delta)\in N$ be such that $x_n\in[1-\delta,1]$ for every $n\ge n_0(\delta)$.Then (3.9.10.1) implies that for every $n\ge n_0(\delta)$

$$|T(x_n,x_{n+1})-T(x_n,1)|=|T(x_n,x_{n+1})-x_n|\le\Psi(x_{n+1})$$

and

$$|T(T(x_n,x_{n+1}),x_{n+2})-T(T(x_n,x_{n+1}),1)|\le\Psi(x_{n+2}).$$

Similarly for every $m\in N$ and every $n\ge n_0(\delta)$

$$\left|\mathrm{T}_{i=n}^{n+m}x_i-T\left(\mathrm{T}_{i=n}^{n+m-1}x_i,1\right)\right|\le\Psi(x_{n+m})$$

and therefore

$$\left|\mathrm{T}_{i=n}^{n+m}x_i-x_n\right|\le\left|\mathrm{T}_{i=n}^{n+m}x_i-\mathrm{T}_{i=n}^{n+m-1}x_i\right|+\ldots.+\left|\mathrm{T}_{i=n}^{n+1}x_i-x_n\right|$$

$$\leq \sum_{i=n+1}^{n+m} \Psi(x_i).$$

If $\sum_{n=1}^{\infty} \Psi(x_n) < \infty$
we obtain that for every $n \geq n_0(\delta)$

$$|\mathrm{T}_{i=n}^{\infty} x_i - x_n| \leq \sum_{i=n+1}^{\infty} \Psi(x_i) \qquad (3.9.10.3)$$

and (3.9.10.3) implies (3.9.10.2).

Corollary 3.9.11 Let T and Ψ be as in Proposition 3.9.10. If for some $q \in (0,1)$,

$$\sum_{n=1}^{\infty} \Psi(1 - q^n) < \infty$$

then T is q-convergent.

Proof: Since $\lim_{n\to\infty} (1 - q^n) = 1$ using 3.9.10.2 we obtain

$$\lim_{n\to\infty} \mathrm{T}_{i=n}^{\infty} (1 - q^n) = \lim_{n\to\infty} (1 - q^n) = 1.$$

Example 3.9.12 Let $\alpha > 0$, $p > 1$ and $h_{\alpha,p} : (0,1] \times [0,1] \to [0,\infty)$ be defined in the following way:

$$h_{\alpha,p}(x,y) = \begin{cases} y - \frac{\alpha}{|\ln(1-x)|^p} & \text{if } (x,y) \in (0,1) \times [0,1], \\ y & \text{if } (x,y) \in \{1\} \times [0,1]. \end{cases}$$

Let f be a continuous function on the closed interval $[a,b]$ and have a derivative at every x in the open interval (a,b). Then there is at least one number c in the open interval (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In this case the function $h_{\alpha,p}$ is equal to zero on the curve which connects the points $(0,1)$ and $1 - e^{-\alpha^{\frac{1}{p}}} < 1$. Let T be a t-norm such that $T(x,y) \geq h_{\alpha,p}(x,y) \quad \forall (x,y) \in [1-\delta, 1] \times [0,1]$. Then $\forall (x,y) \in [0,1] \times [1-\delta, 1)$

$$\begin{aligned} |T(x,y) - T(x,1)| &= |T(y,x) - T(1,x)| \\ &\leq |h_{\alpha,p}(x,y) - h_{\alpha,p}(1,x)| \\ &= \left| x - \frac{\alpha}{|\ln(1-y)|^p} - x \right| \\ &\leq \frac{\alpha}{|\ln(1-y)|^p}, \end{aligned}$$

i.e., (3.9.10.1) holds for

$$\Psi(y) = \begin{cases} \frac{\alpha}{|\ln(1-y)|^p} & \text{if } y \in [1-\delta, 1), \\ 0 & \text{if } y = 1. \end{cases}$$

Since

$$\begin{aligned} \sum_{n=1}^{\infty} \Psi(1-q^n) &= \sum_{n=1}^{\infty} \frac{\alpha}{|\ln(q^n)|^p} \\ &= \sum_{n=1}^{\infty} \frac{\alpha}{n^p |\ln(q)|^p} < \infty, \end{aligned}$$

which shows that T is q-convergent.

CHAPTER 4

SOME SELECTED FIXED POINT THEOREMS IN PM-SPACES

4.1 Introduction

Fixed point theory in PM-spaces is discussed which is a part of Probabilistic Analysis and continues to be an active area of mathematical research. Fixed point theory in PM-spaces is still in its formative stage. We anticipate that in near future we shall witness the development of a unified theory of random equations that will be based, partly, on concepts and tools of the fixed point theory in PM-spaces. Sherwood [54] showed that for a very large class of triangular norms it is possible to construct complete Menger PM-spaces admitting contraction mappings without fixed points. Recently, Hicks [19] considered another notion of contraction mappings namely H-contractions and showed that every such a mapping on a complete Menger PM-space $(X, \mathcal{F}, \min)$ has a unique fixed point. We conclude this chapter with a relatively recent fixed point theorem due Singh and Jain [58] which generalizes some earlier results due to Mishra [30], Hadžić [16] and Singh and Pant [59].

In this chapter, we shall present some important but selected results for fixed point theorem on PM-spaces.

4.2 Contraction mapping in PM-spaces

The following definition of a contraction mapping was suggested and studied by Sehgal and Bharucha-Reid [50], which is a very natural probabilistic version of the notion of strict contraction in metric spaces.

Definition 4.2.1 A mapping $T : (X, \mathcal{F}) \longrightarrow (X, \mathcal{F})$ is said to be a contraction mapping (or SB-contraction mapping) if and only if there is an $\alpha \in (0, 1)$ such that

$$F_{Tp,Tq}(t) \geq F_{p,q}\left(\frac{t}{\alpha}\right), \quad \forall p, q \in X \text{ and } t > 0. \tag{4.2.1.1}$$

Theorem 4.2.2 Let Δ be lc-t-norm and Δ_T be the lc-t-function defined by 1.2.9.1. Then there exists a complete PM-space $(X, \mathcal{F})$ satisfying PM-6 under Δ_T and a contraction map T on $(X, \mathcal{F})$ which has no fixed point if and only if there exists a $G \in D$ and a number $\alpha \in (0, 1)$ such that

$$\sup\left\{\tau_{i=1}^{\infty}\, G\left(\frac{j}{\alpha^{i-1}}\right)(x) : x \in R\right\} < 1,$$

where $j : R \to R$ is the identity function.

Proof: The proof is very long and involved, therefore it is omitted. For the proof one may consult Chang et. al [9].

The following two consequences of Theorem 4.2.2 are worth noting.

Corollary 4.2.3 There is a complete PM-space $(X, \mathcal{F})$ (satisfying PM-5 under Δ_m) admitting a contraction mapping without fixed point.

Proof: Let α be any number with $0 < \alpha < 1$ and let G be defined via

$$G(t) = \left\{ \begin{array}{ll} 0, & t \leq \frac{1}{\alpha^2}. \\ 1 - \frac{1}{n}, & \frac{1}{\alpha^2} < t \leq \frac{1}{\alpha^{n+1}} \end{array} \right\}, \quad n > 1.$$

It is easily verified in view of Theorem 4.2.2.

Corollary 4.2.4 There is a complete PM-space $(X, \mathcal{F})$ satisfying PM-5 under $\Delta_2 =$ product norm and a contraction mapping on $(X, \mathcal{F})$ having no fixed point.

Proof: The earlier distribution function G used by Corollary 4.2.3 again serve our purpose.

Definition 4.2.5 Let Δ be a t-norm satisfying condition $\sup\limits_{0<t<1} \Delta(t,t) = 1$. Δ is said to be of h-type if the family of functions $\{\Delta^m(t)\}_{m=1}^{\infty}$
is equi-continuous at $t = 1$, where $\Delta'(t) = \Delta(t,t)$, $\Delta^m(t) = \Delta(t, \Delta^{m-1}(t))$, $t \in [0,1]$, $m = 2, 3, 4$.

Definition 4.2.6 A t-norm Δ is said to have the fixed point property (f.p.p.) if every contraction mapping on a complete Menger PM-space has a fixed point.

Lemma 4.2.7 If for every contraction mapping T on a Menger PM-space $(X, \mathcal{F}, \Delta)$ and for each given point $p_0 \in X$, the sequence $\{p_n\}$ defined by $p_n = T^n p_0$ is a Δ_T -Cauchy sequence, then Δ has a fixed point property.

Proof: Due to a limited space, we skip its proof be referred to Chang et. al [9].

Theorem 4.2.8 Every t-norm of h-type has the fixed point property.

Proof: Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space such that Δ is of h-type and $T : X \longrightarrow X$ be a contraction mapping satisfying the following $F_{Tx,\,Ty}(kt) \geq F_{x,y}(t)\,,\, x, y \in X,\ t \geq 0$, where $k \in (0,1)$ is a constant (For the sake of convenience, we only consider $k \in (0, \frac{1}{2})$. The proof is similar if $k = \left[\frac{1}{2}, 1\right)$,).
Let $x_0 \in X$ be any given point. Then for any positive integer m and $t \in R^+$, we have

$$\begin{aligned} F_{x_0,\,T_{x_0}^{m=1}}(2t) &\geq \Delta\left(F_{x_0,Tx_0}(t)\,,\, F_{Tx_0,\,T_{x_0}^{m=1}}(t)\right) \\ &\geq \Delta\left(F_{x_0,Tx_0}(t)\,,\, F_{x_0,\,T_{x_0}^{m}}(2t)\right) \\ &\geq \ldots\ldots\ldots\ldots\ldots\ldots \\ &\geq \Delta^m\left(F_{x_0,\,T_{x_0}}(t)\right). \end{aligned}$$

Hence for any positive integers n, m, we have

$$F_{T^n x_0,\,T^{n+m}x_0}(2t) \geq \Delta^m\left(F_{x_0,\,Tx_0}\left(k^{-n}t\right)\right).$$

Since Δ is of h-type, it follows that $\lim_{n\to\infty} F_{T^n x_0,\,T^{n+m}x_0}(2t) = 1,\ t > 0$, uniformly in m. This implies that $\left\{T_{x_0}^n\right\}$ is a Cauchy sequence in X.

Theorem 4.2.9 Let Δ be a lc-t-norm with the fixed point property. Then Δ is of h-type.

Proof: Suppose on contrary that Δ is not of h-type, then there exists $t_0\ \varepsilon\ (0,1)$ such that for any $t > t_0$ there exists a positive integer $m = m(t)$ for which $\Delta^m(t) < t_0$. Now let $t_n \in (t_0, 1)$ be increasing to 1. Then there exists positive integer m_n such that

$$\Delta^{m_n}(t_n) < t_0, n = 1, 2, 3, \ldots\ldots$$

Obviously, we can suppose that m_n is increasing.
Let $F \in D$ be defined by

$$F(t) = \left\{\begin{array}{lr} 0 & t \leq 1, \\ t_1 & t \in (1, 2^{2+m_1}]\,, \\ t_{n+1} & t \in (2^{2n+m_n}, 2^{2n+2+m_n+1}] \end{array}\ ,\ n \geq 1.\right\}$$

Using F, we define a probabilistic metric $F : \mathbb{N} \times \mathbb{N} \longrightarrow D$ by

$$F_{n,n+m}(t) = \Delta\left(F\left(2^{n+1}t\right), \Delta\left(F\left(2^{n+2}t\ ,\Delta(\ldots\ldots.,\Delta\left(F\left(2^{n+m-1}t\right), F\left(2^{n+m}t\right)\right)\right)\right)\right),$$

$$F_{n,n}(t) = H(t).$$

It is easy to show that ($\mathbb{N}$, $\mathcal{F}$, Δ) is a Menger PM-space and the mapping $T : \mathbb{N} \longrightarrow \mathbb{N}$. $T(n) = n+1$ is a contraction mapping with the Lipschitz constant $\frac{1}{2}$. Therefore we have

$$\begin{aligned} F_{T^{n-1}(1),\, T^{n-1+m_n}(1)}(1) &= F_{n,n+m_n}(1) \leq F_{n,n+m_n}(2^n) \\ &= \Delta\left(F\left(2^{n+1}\right), \Delta\left(F\left(2^{n+2}\right), \ldots\ldots, \Delta\left(F2^{2n+m_n-1}, F\left(2^{2n+m_n}\right)\right)\right)\right), \\ &\leq \Delta^{2n+m_n-1}\left(F\left(2^{2n+m_n}\right)\right) \\ &\leq \Delta^{2n+m_n-1}(t_n) < t_0 \end{aligned}$$

which implies that the sequence $\{T^n(1)\}$ is not a Cauchy sequence. From Lemma 4.2.7, Δ has no fixed point property, which is a contradiction.

Theorem 4.2.10 In respect of a continuous t-norm Δ, the following are equivalent:
(i) Δ has the fixed point property
(ii) Δ is of h-type.

Remark 4.2.11 It should pointed out that theorem is essentially equivalent to the classical Banach contraction principle and the t-norm of h-type is essentially the largest one for which the Banach contraction principle generally holds.

Corollary 4.2.12 [54] If Δ is an Archimedian t-norm, then Δ does not have the f.p.p.

Proof: Since Δ is an Archimedian t- norm, Δ is continuous and has no idempotents.

Recently, R. M. Tardiff [62] proved the following result.

Theorem 4.2.13 Let $(X, \mathcal{F}, \Delta)$ be a complete Menger PM-space. Suppose that Δ is continuous and $\Delta \geq \Delta_m$. If for any $p, q \in X$,

$$\int_1^\infty I_n(t)\, dF_{p,q}(t) < \infty,$$

then any contraction mapping on $(X, \mathcal{F}, \Delta)$ has a unique fixed point.

Proof: For the proof one may consult Chang et. al [9].

4.3 Hicks Fixed Point Theorems and Its Generalizations

In 1983, Hicks [19] introduced another notion of contraction mappings, which referred is generally as H-contraction.

Definition 4.3.1 Let $(X, \mathcal{F}, \Delta)$ be a H-contraction mapping, if for any $t > 0$ and for any $x, y \in X$,

$$F_{Tx,\, Ty}(kt) > 1 - kt \tag{4.3.1.1}$$

whenever $F_{x,y}(t) > 1 - t$, where $k \in (0,1)$ is a constant.

Lemma 4.3.2 Let $(X, \mathcal{F}, \Delta)$ be a complete Menger PM-space, $T : X \longrightarrow X$ be a H-contraction mapping satisfying (4.3.1.1). Then
(i) For any $\epsilon > 0$, $\lambda \in (0,1)$, there exists a positive integer $n(\epsilon, \lambda)$ such that, for all $p, q \in X$ and $n \geq n(\epsilon, \lambda)$,

$$F_{T^n p\, T^n q}(\epsilon) > 1 - \lambda, \tag{4.3.2.1}$$

(ii) T has atmost one fixed point in X,
(iii) T is uniformly continuous.

Proof: (i) Since clearly $F_{p,q}(1+\epsilon) > 1 - (1+\epsilon)$, it follows from equation 4.3.1.1 that

$$F_{T^n p, T^n q}(k^n(1+\epsilon)) > 1 - k^n(1+\epsilon).$$

Since $k \in (0,1)$, For any $\epsilon > 0$, $\lambda \in (0,1)$, there exists $n(\epsilon, \lambda) \in \mathbb{N}$, such that k^n $(1+\epsilon) \leq min\ \{\epsilon, \lambda\}$, for any $n \geq n(\epsilon, \lambda)$.
Hence for any $n \geq n(\epsilon, \lambda)$, we have

$$F_{T^n p,\, T^n q}(\epsilon) \geq F_{T^n p, T^n q}(k^n(1+\epsilon)) > 1 - k^n(1+\epsilon) \geq 1 - \lambda.$$

(ii) If $p, q \in X$ are two fixed points of T, then for any $n \geq 1$ and $T^n p = p, T^n q = q$.
By the conclusion 4.3.1.1,
For any $\epsilon > 0$ and for any $\lambda \in (0,1)$, $F_{p,q}(\epsilon) > 1 - \lambda$, which implies that $p = q$.
(iii) Let $\epsilon > 0$ and $\lambda \,\varepsilon\, (0,1)$ be given and choose $\delta > 0$ such that $k\delta < min\ \{\epsilon, \lambda\}$. Now if $p, q \in N(\delta, \delta)$, where $N(\delta, \delta)$ is the $(\delta, \delta)-$ neighborhood, i.e.,$F_{p,q}(\delta) > 1 - \delta$.
Since T is a H-contraction mapping, we have

$$F_{Tp,Tq}(k\delta) \geq 1 - k\delta.$$

Hence, it follows that

$$F_{Tp,Tq}(\epsilon) \geq F_{Tp,Tq}(k\delta) > 1 - k\delta > 1 - \lambda,$$

which means that

$$(Tp, Tq) \in N(\epsilon, \lambda).$$

Theorem 4.3.3 Let $(X, \mathcal{F}, \Delta)$ be a complete Menger PM-space such that

$$\sup_{0<t<1} \Delta(t,t) = 1.$$

Then each H-contraction mapping T on X has a unique fixed point and, for any given $p_0 \in X$, the iterative sequence $\{T^n p_0\}$ converges to this fixed point.

Proof: Let $\epsilon > 0$, $\lambda \in (0,1)$ be given. Then by Lemma 4.3.2, there exists a positive integer $n(\epsilon, \lambda)$ such that (4.3.2.1) holds.
Taking $p = p_m$ and $q = p_0$, then for all $n \geq n(\epsilon, \lambda)$ and $m \geq 1$, we have

$$F_{p_{n+m}, p_n}(\epsilon) = F_{T^n p_m,\, T^n p_0}(\epsilon) > 1 - \lambda.$$

Therefore, $\{p_n\}$ is a Cauchy sequence. Since X is complete, we may assume that $p_n \longrightarrow p^*$. By Lemma 4.3.2, it follows that $T_{p^*} = p^*$, i.e., p^* is a fixed point of T and it is unique, which completes the proof.

If the t-norm Δ in Theorem 4.3.3 satisfies the following condition

$$\Delta(a, b) \geq max\{a + b - 1, 0\}, \ a, b \in [0, 1], \tag{4.3.3.1}$$

then we have the following:

Theorem 4.3.4 Let $(X, \mathcal{F}, \Delta)$ be a complete Menger PM-space and Δ be a t-norm satisfying (4.3.3.1), then the mapping $T : X \longrightarrow X$ is a H-contraction on $(X, \mathcal{F}, \Delta)$ if and only if T is a Banach contraction mapping on a metric space (X, d^*), i.e., if and only if there exists a $k \in (0, 1)$ such that

$$d^*(Tx,\, Ty) \leq kd^*(x, y). \tag{4.3.4.1}$$

Proof: First we point out that the function d^* defined as

$$d^*(x, y) = \sup_t \left\{t \in R^+ : F_{x,y}(t) > 1 - t\right\} \tag{4.3.4.2}$$

has the following property :

$$d^*(x, y) < t \Longleftrightarrow F_{x,y}(t) > 1 - t, \ t > 0. \tag{4.3.4.3}$$

If $T : X \longrightarrow X$ is a H-contraction on $(X, \mathcal{F}, \Delta)$, then for any $t > 0$ such that $F_{x,y} > 1 - t$, we have

$$F_{Tx,Ty}(kt) > 1 - kt.$$

By equation (4.3.4.3), that is to say for any $t > 0$, if $d^*(x, y) < t$, then we have $d^*(Tx,\, Ty) < kt$. Let $t \rightarrow d^*(x, y)$, for all $x, y \in X$, we have

$$d^*(Tx, Ty) \leq k.d^*(x, y). \tag{4.3.4.4}$$

This shows that T is a Banach contraction mapping on (X, d^*).

Conversely, if T is a Banach contraction mapping on (X, d^*) which satisfies (4.3.4.4), then for any $t > 0$ such that $F_{x,y}(t) > 1 - t$.

By equation (4.3.4.3), we have $d^*(x, y) \leq t$.
From equation (4.3.4.4), it follows that $d^*(Tx, Ty) < kt$.
Hence

$$F_{Tx,Ty}(kt) > 1 - kt,$$

which shows that T is a H-contraction.

Definition 4.3.5 Two self maps A and S of a Menger space $(X, \mathcal{F}, \Delta)$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e., if $Ap = Sp$ for some $p \in X$ then $ASp = SAp$.

Definition 4.3.6 [30] Two self maps A and S of a Menger space $(X, \mathcal{F}, \Delta)$ are called compatible if $\forall x > 0$

$$F_{ASp_n, SAp_n}(x) \to 1$$

whenever $\{p_n\}$ is a sequence in X such that Ap_n, $Sp_n \to u$, for some u in X, as $n \to \infty$.

Proposition 4.3.7 A pair of self mappings A and S of a Menger space $(X, \mathcal{F}, \Delta)$ is compatible then the pair is also weakly compatible.

Proof: Suppose $Ap = Sp$, for some p in X. Consider the constant sequence $\{p_n\} = p$. Now, $\{Ap_n\} \to Ap$ and $\{Sp_n\} \to Sp\ (= Ap)$. As A and S are compatible we have

$$F_{ASp_n, SAp_n}(x) \to 1 \quad \forall x > 0.$$

Thus, $ASp = SAp$ and we get that the pair (A, S) is weakly compatible.

Example 4.3.8 Let (X, d) be a metric space where $X = [0, 2]$ and $(X, \mathcal{F}, \Delta)$ be the induced Menger space with $F_{p,q}(\epsilon) = H(\epsilon - d(p, q))$, $\forall p, q \in X$ and $\forall \epsilon > 0$. Define self maps A and S as follows:

$$Ax = \begin{cases} 2 - x & if\ 0 \leq x < 1, \\ 2 & if\ 1 \leq x \leq 2, \end{cases} \text{ and } Sx = \begin{cases} x & if\ 0 \leq x < 1, \\ 2 & if\ 1 \leq x \leq 2. \end{cases}$$

Take $x_n = 1 - \frac{1}{n}$. Now,

$$F_{Ax_n,1}(\epsilon) = H\left(\epsilon - \frac{1}{n}\right) \qquad so,\ \lim_{n\to\infty} F_{Ax_n,1}(\epsilon) = H(\epsilon) = 1.$$

Hence $Ax_n \to 1$ as $n \to \infty$. Similarly, $Sx_n \to 1$ as $n \to \infty$. Also

$$F_{ASx_n, SAx_n}(\epsilon) = H\left(\epsilon - \left(1 - \frac{1}{n}\right)\right), \ \lim_{n\to\infty} F_{ASp_n, SAp_n}(\epsilon) = H(\epsilon - 1) \neq 1, \forall \epsilon > 0.$$

Hence the pair (A, S) is not compatible. Also set of coincidence points of A and S is $[1,2]$. Now for any $x \in [1,2]$, also $Ax = Sx = 2$, and $AS(x) = A(2) = 2 = S(2) = SA(x)$. Thus A and S are weakly compatible but not compatible.

Lemma 4.3.9 [59] Let $\{x_n\}$ be a sequence in a Menger space $(X, \mathcal{F}, \Delta)$ with continuous Δ-norm and $\Delta(x, x) \geq x$. Suppose, for all $x \in [0,1]$, $\exists k \in (0,1)$ such that $\forall x > 0$ and $n \in N$,

$$F_{x_n, x_{n+1}}(kx) \geq F_{x_{n-1}, x_n}(x).$$

Then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 4.3.10 Let $(X, \mathcal{F}, \Delta)$ be a Menger space. If there exists $k \in (0,1)$ such that for $p, q \in X$, $F_{p,q}(kx) \geq F_{p,q}(x)$. Then $p = q$.

Proof: As $F_{p,q}(kx) \geq F_{p,q}(x)$, we have

$$F_{p,q}(x) \geq F_{p,q}\left(k^{-1}x\right).$$

By repeated application of above inequality, we get

$$F_{p,q}(x) \geq F_{p,q}\left(k^{-1}x\right) \geq F_{p,q}\left(k^{-2}x\right) \geq \ldots\ldots \geq F_{p,q}\left(k^{-m}x\right) \geq \ldots., \quad m \in N$$

which tends tol as $m \to \infty$. Hence, $F_{p,q}(x) = 1$, $\forall x > 0$ and we get $p = q$.

Now we are fully equipped to state and prove the following recent fixed point theorem due to Singh and Jain [58].

Theorem 4.3.11 Let A, B, S, T, L and M are self maps on a complete Menger space, $(X, \mathcal{F}, \Delta)$ with $t(a, a) \geq a$, $\forall a\varepsilon\ [0,1]$, satisfying
(i) $L(X) \subseteq ST(X)$, $M(X) \subseteq AB(X)$,
(ii) $AB = BA$, $ST = TS$, $LB = BL$, $MT = TM$,
(iii) either AB or L is continuous,
(iv) (L, AB) is compatible and (M, ST) is weakly compatible,
(v) there exists $k \in (0,1)$ such that

$$F_{Lp,Mq}(kx) \geq \min\left\{F_{ABp,Lp}(x), F_{STq,Mq}(x), F_{STq,Lp}(\beta x), F_{ABp,Mq}((2-\beta)x), F_{ABp,STq}(x)\right\}$$

$\forall p, q \in X$, $\beta \in (0,2)$ and $x > 0$.
Then A, B, S, T, L and M have a unique common fixed point in X.

Proof: Let $x_0 \in X$. From condition (i) there exists $x_1, x_2 \in X$ such that $Lx_0 = STx_1 = y_0$ and $Mx_1 = ABx_2 = y_1$. Inductively we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $Lx_{2n} = STx_{2n+1} = y_{2n}$ and $Mx_{2n+1} = ABx_{2n+2} = y_{2n+1}$ for $n = 0, 1, 2, 3, \ldots.$:

By writing $p = x_{2n},\ q = x_{2n+1}$ for $x > 0$ and $\beta = 1 - q$ with $q \in (0,1)$ in (v), we get
$F_{Lx_{2n},Mx_{2n+1}}(kx) \geq \min\{F_{ABx_{2n},Lx_{2n}}(x), F_{STx_{2n+1},Mx_{2n+1}}(x), F_{STx_{2n+1},Lx_{2n}}((1-q)x),$
$F_{ABx_{2n},Mx_{2n+1}}((1+q)x), F_{ABx_{2n},STx_{2n+1}}(x)\}$

$$
\begin{aligned}
&F_{y_{2n},y_{2n+1}}(kx)\\
&\quad\geq \min\left\{F_{y_{2n-1},y_{2n}}(x), F_{y_{2n},y_{2n+1}}(x), 1, F_{y_{2n-1},y_{2n+1}}((1+q)x), F_{y_{2n-1},y_{2n}}(x)\right\}\\
&\quad\geq \min\left\{F_{y_{2n-1},y_{2n}}(x), F_{y_{2n},y_{2n+1}}(x), F_{y_{2n-1},y_{2n}}(x), F_{y_{2n},y_{2n+1}}(qx)\right\}\\
&\quad\geq \min\left\{F_{y_{2n-1},y_{2n}}(x), F_{y_{2n},y_{2n+1}}(x), F_{y_{2n},y_{2n+1}}(qx)\right\}.
\end{aligned}
$$

As Δ-norm is continuous, letting $q \to 1$ we get

$$
\begin{aligned}
F_{y_{2n},y_{2n+1}}(kx) &\geq \min\left\{F_{y_{2n-1},y_{2n}}(x), F_{y_{2n},y_{2n+1}}(x), F_{y_{2n},y_{2n+1}}(x)\right\}\\
&= \min\left\{F_{y_{2n-1},y_{2n}}(x), F_{y_{2n},y_{2n+1}}(x)\right\},
\end{aligned}
$$

implying thereby $F_{y_{2n},y_{2n+1}}(kx) \geq \min\left\{F_{y_{2n-1},y_{2n}}(x), F_{y_{2n},y_{2n+1}}(x)\right\}$.
Similarly, one can show that $F_{y_{2n+1},y_{2n+2}}(kx) \geq \min\left\{F_{y_{2n},y_{2n+1}}(x), F_{y_{2n+1},y_{2n+2}}(x)\right\}$. Therefore, for all n even or odd we have

$$F_{y_n,y_{n+1}}(kx) \geq \min\left\{F_{y_{n-1},n}(x), F_{y_n,y_{n+1}}(x)\right\}.$$

Which in turn yields

$$F_{y_n,y_{n+1}}(x) \geq \min\left\{F_{y_{n-1},n}\left(k^{-1}x\right), F_{y_n,y_{n+1}}\left(k^{-1}x\right)\right\}.$$

By repeated application of above inequality, one gets

$$F_{y_n,y_{n+1}}(x) \geq \min\left\{F_{y_{n-1},n}\left(k^{-1}x\right), F_{y_n,y_{n+1}}\left(k^{-m}x\right)\right\}.$$

Since $F_{y_n,y_{n+1}}(k^{-m}x) \to 1$ as $m \to \infty$, it follows that

$$F_{y_n,y_{n+1}}(kx) \geq F_{y_{n-1},n}(x) \quad \forall\, n \in N \text{ and } \forall\, x > 0.$$

Therefore, by Lemma 4.3.9, $\{y_n\}$ is a Cauchy sequence in X which is due to completeness of X, converges in X, i.e. $\{y_n\} \to z \in X$. Consequently its subsequences also converge as follows:

$$\{Mx_{2n+1}\} \to z \text{ and } \{STx_{2n+1}\} \to z, \tag{4.3.11.1}$$

$$\{Lx_{2n}\} \to z \text{ and } \{ABx_{2n}\} \to z. \tag{4.3.11.2}$$

If AB is continuous, then $(AB)^2 x_{2n} \to ABz$ and $(AB)Lx_{2n} \to ABz$. As (L, AB) is compatible, we have $L(AB)x_{2n} \to ABz$. Now setting firstly $p = ABx_{2n}, q = x_{2n+1}$ with $\beta = 1$ in condition (v), we get

$F_{ABx_{2n},Mx_{2n+1}}(kx) \geq \min\{ F_{ABABx_{2n},LABx_{2n}}(x), F_{STx_{2n+1},Mx_{2n+1}}(x), F_{STx_{2n+1},LABx_{2n}}(x),$
$F_{ABABx_{2n},Mx_{2n+1}}(x), F_{ABABx_{2n},STx_{2n+1}}(x)\},$

which on letting $n \to \infty$ reduces to

$$F_{ABz,z}(kx) \geq \min\{F_{ABz,ABz}(x), F_{z,z}(x), F_{z,ABz}(x), F_{ABz,z}(x), F_{ABz,z}(x)\},$$

implying thereby $F_{ABz,z}(kx) \geq F_{ABz,z}(x)$. Therefore, by Lemma 4.3.10, we get

$$ABz = z. \tag{4.3.11.3}$$

Now setting secondly $p = z, q = x_{2n+1}$ with $\beta = 1$ in condition (v), we get
$F_{Lz,Mx_{2n+1}}(kx) \geq \min\{F_{ABz,Lz}(x), F_{STx_{2n+1},Mx_{2n+1}}(x),\ F_{STx_{2n+1},Lz}(x),$
$F_{ABz,Mx_{2n+1}}(x), F_{ABz,STx_{2n+1}}(x)\},$

which on letting $n \to \infty$, reduces to

$$F_{Lz,z}(kx) \geq min\{F_{z,Lz}(x), F_{z,z}(x), F_{z,Lz}(x), F_{Lz,z}(x), F_{Lz,z}(x)\},$$

yielding thereby $Lz = z$. Therefore $ABz = Lz = z$.

Thirdly, setting $p = Bz, q = x_{2n+1}$ with $\beta = 1$ in condition (v), we get
$F_{LBz,Mx_{2n+1}}(kx) \geq \min\{F_{ABBz,LBz}(x), F_{STx_{2n+1},Mx_{2n+1}}(x),\ F_{STx_{2n+1},LBz}(x),$
$F_{ABBz,Mx_{2n+1}}(x), F_{ABBz,STx_{2n+1}}(x)\}.$

As $BL = LB, AB = BA$, so we have $L(Bz) = B(Lz) = Bz$ and $AB(Bz) = B(ABz) = Bz$. Letting $n \to \infty$, we get

$$F_{Bz,z}(kx) \geq min\{F_{Bz,Bz}(x), F_{z,z}(x), F_{z,Bz}(x), F_{Bz,z}(x), F_{Bz,z}(x)\},$$

i.e. $F_{Bz,z}(kx) \geq F_{Bz,z}(x)$; which gives $Bz = z$ and $ABz = z$ implies $Az = z$. Therefore

$$Az = Bz = Lz = z. \tag{4.3.11.4}$$

As $L(X) \subseteq ST(X)$, there exists $v \in X$ such that $z = Lz = STv$. Setting $p = x_{2n}, q = v$ with $\beta = 1$ in condition (v), we get
$F_{Lx_{2n},Mv}(kx) \geq \min\{F_{ABx_{2n},Lx_{2n}}(x), F_{STv,Mv}(x),\ F_{STv,Lx_{2n}}(x),$
$F_{ABx_{2n},Mv}(x), F_{ABx_{2n},STv}(x)\}.$

Now, letting $n \to \infty$ and using equation (4.3.11.2), we get

$$F_{z,Mv}(kx) \geq min\{F_{z,z}(x), F_{z,Mv}(x), F_{z,z}(x), F_{z,Mz}(x), F_{z,z}(x)\} \geq F_{z,Mv}(x).$$

Therefore by Lemma 4.3.10, $Mv = z$. Hence, $STv = z = Mv$. As (M, ST) is weakly compatible, we have $STMv = MSTv$. Thus, $STz = Mz$.

In order to prove that $Mz = z$, we set $p = x_{2n}, q = z$ with $\beta = 1$ in condition (v), we get
$F_{Lx_{2n},Mz}(kx) \geq \min\{F_{ABx_{2n},Lx_{2n}}(x), F_{STz,Mz}(x), F_{STz,Lx_{2n}}(x),$
$F_{ABx_{2n},Mz}(x), F_{ABx_{2n},STz}(x)\}$.

Now, letting $n \to \infty$ and using equation (4.3.11.1), we get

$$F_{z,Mz}(kx) \geq min\{F_{z,z}(x), F_{Mz,Mz}(x), F_{Mz,z}(x), F_{z,Mz}(x), F_{z,Mz}(x)\},$$

i.e. $F_{z,Mz}(kx) \geq F_{z,Mz}(x)$, implying thereby $z = Mz$.

For proving $Tz = z$, we set $p = x_{2n}, q = Tz$ with $\beta = 1$ in condition (v), we get
$F_{Lx_{2n},MTz}(kx) \geq \min\{F_{ABx_{2n},Lx_{2n}}(x), F_{STTz,MTz}(x), F_{STTz,Lx_{2n}}(x),$
$F_{ABx_{2n},MTz}(x), F_{ABx_{2n},STTz}(x)\}$.

Making use of $MT = TM$ and $ST = TS$ we have $MTz = TMz = Tz$ and $ST(Tz) = T(STz) = Tz$.
and letting $n \to \infty$, we get

$$F_{z,Tz}(kx) \geq min\{F_{z,z}(x), F_{Tz,Tz}(x), F_{Tz,z}(x), F_{z,Tz}(x), F_{z,Tz}(x)\}.$$

Which in view of Lemma 4.3.10, yields $Tz = z$.
As $STz = Tz = z$ implies $Sz = z$, therefore

$$Sz = Tz = Mz = z. \tag{4.3.11.5}$$

Combining (4.3.11.4) and (4.3.11.5), we get $Az = Bz = Lz = Mz = Tz = Sz = z$. Hence, the six self maps have a common fixed point in this case.

In case L is continuous. $L^2x_{2n} \to Lz$ and $L(ABx_{2n}) \to Lz$. As (L, AB) is compatible, we have $(AB)Lx_{2n} \to Lz$.

By setting $p = Lx_{2n}, q = x_{2n+1}$ with $\beta = 1$ in condition (v), we get

$F_{LLx_{2n},Mx_{2n+1}}(kx) \geq min\{F_{ABLx_{2n},LLx_{2n}}(x), F_{STx_{2n+1},Mx_{2n+1}}(x), F_{STx_{2n+1},LLx_{2n}}(x),$
$F_{ABLx_{2n},Mx_{2n+1}}(x), F_{ABLx_{2n},STx_{2n+1}}(x)\}$,

which on letting $n \to \infty$ reduces

$$F_{Lz,z}(kx) \geq min\{F_{Lz,Lz}(x), F_{z,z}(x), F_{z,Lz}(x), F_{Lz,z}(x), F_{Lz,z}(x)\},$$

yielding thereby $Lz = z$.

As $M(X) \subseteq AB(X)$ there exists $w \in X$ such that $z = Mz = ABw$. Setting $p = w, q = x_{2n+1}$ with $\beta = 1$ in condition (v), we get
$F_{Lw,Mx_{2n+1}}(kx) \geq \min\{F_{ABw,Lw}(x), F_{STx_{2n+1},Mx_{2n+1}}(x), F_{STx_{2n+1},Lw}(x),$
$F_{ABw,Mx_{2n+1}}(x), F_{ABw,STx_{2n+1}}(x)\}.$

which on letting $n \to \infty$ reduces to

$$F_{Lw,z}(kx) \geq min\{F_{z,Lw}(x), F_{z,z}(x), F_{z,Lw}(x), F_{z,z}(x), F_{z,z}(x)\},$$

i.e. $F_{Lw,z}(kx) \geq F_{z,Lw}(x)$, which gives $Lw = z$. As (L, AB) is weakly compatible, we have $Lz = ABz$ which together with $Bz = z$ gives that $Az = Bz = Lz = z$ and we find that z is the common fixed point of the six maps in this case also.

To prove the uniqueness, let u be another common fixed point of A, B, L, M, S and T, then

$$Au = Bu = Lu = Su = Tu = Mu = u.$$

Putting $p = z, q = u$ with $\beta = 1$ in condition (v), we get

$$F_{Lz,Mu}(kx) \geq min\{F_{ABz,Lz}(x), F_{STu,Mu}(x), F_{STu,Lz}(x), F_{ABz,Mu}(x), F_{ABz,STu}(x)\},$$

i.e. $F_{z,u}(kx) \geq F_{z,u}(x)$, which gives $z = u$. Therefore, z is a unique common fixed point of A, B, L, M, S and T. This completes the proof.

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